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Mathematical Reviews

Vol. 15, No. 10 NOVEMBER, 1954 Pages 845-922

FOUNDATIONS

Quine, Willard Van Orman. **From a logical point of view. 9 logicophilosophical essays.** Harvard University Press, Cambridge, Mass. 1953. vii+184 pp. $3.50.

This book contains both previously published and new material. However, the author has not restricted himself to having the older parts simply reprinted. He has done a considerable amount of editing, and in the case of his well-known paper on "New foundations" [Amer. Math. Monthly 44, 70–80 (1937)] he has added valuable "Supplementary remarks". As a result, we have a very helpful volume which sums up Quine's ideas and contributions on semantics, nominalism, and foundations of logic, and which from the importance of these subjects and from Quine's well-deserved influence derives a special interest.

The presentation is self-contained and lucid. In the interest of many readers, the following observations seem, however, justified. The treatment of descriptions (pp. 85 ff.) leaves one in the dark as to the conditions for the introduction of a description. This point has been cleared up by Rosser [for instance, in his "Logic for mathematicians", McGraw-Hill, New York, 1953, pp. 212 ff.; these Rev. 14, 935], but his work on the subject is not mentioned. Also, no mention is made of Gödel's penetrating essay on "Russell's mathematical logic" [P. A. Schilpp (ed.), The philosophy of Bertrand Russell, Northwestern Univ., Evanston, Illinois, 1944, pp. 125–153], which is of exceptional importance in connection with discussions on nominalism and platonism.

A leaflet, apparently not distributed with all copies, mentions the following corrections: p. 118, l. 7 above footnote, for "abbreviations respectively" read "the conjunction and alternation"; p. 125, for lines 17 and 18, read: "where h is $m-1$ and k is $m+1$. We then also need a postulate '$x=y\supset (x \in z\equiv y \in z)$', for all choices of exponents"; p. 180, under Fitch, for: 176, read: 177, and under Henkin, for: 172, read: 173; p. 182, under Nelson, for: 169, read: 170.

E. W. Beth (Amsterdam).

Reichenbach, Hans. **Nomological statements and admissible operations.** Studies in logic and the foundations of mathematics. North-Holland Publishing Co., Amsterdam, 1954. 140 pp. 13.50 florins.

The book contains the development of a theory known from the "Elements of symbolic logic" of the same author [Macmillan, New York, 1947; these Rev. 8, 557]. Its principal aim is the "explication" of some logical terms used in physical science in a meaning different from the one they have in extensional mathematical logic. The explication concerns "connective operations", e.g., $a\supset b$ in the case a, b are statements with physical terms. Such methodological terms as "true", "verifiable", "synthetic" are considered as known to the reader. There is a gradation of true statements: (1) tautologies, (2) synthetic nomological statements, admissible or not admissible, (3) not nomological but verifiably true, (4) not verifiably true but factually true, permissible or not. Original nomological statements are defined by means of nine conditions; by definition, they must be verifiably true and exhaustive. The last condition expresses in short that every cancellation of a logical summand in the normal form gives a statement which is not verifiably true. Other nomological statements are defined as being deductive consequences of the original ones. Nomological statements satisfying some weaker condition of exhaustiveness (quasi-exhaustive) are called admissible. In chapter VII, "Counterfactuals of noninterference", the probability $P(b)$ and the relative probability $P(a, b)$ are used. If a is false and b true, then the counterfactual implication $a\supset b$ is called permissible if and only if a does not interfere with b, i.e. if $P(a, b)\geq P(b)$. A few remarks are made about the limitations of known logical theorems, e.g., of the law of contraposition when applied to connective operations. The author does not discuss any method of discovering whether a logical law remains valid for connective operations or not. His results have an importance rather for the logic of empirical science, though they suggest some new problems for mathematical logic.

S. Jaśkowski (Toruń).

Łoś, J. **On the categoricity in power of elementary deductive systems and some related problems.** Colloquium Math. 3, 58–62 (1954).

Given a cardinal number $\mathfrak{m}$ the author defines a deductive system to be categorical in the power $\mathfrak{m}$ if and only if it possesses only one model of cardinal number $\mathfrak{m}$. He gives examples to illustrate this notion and then states some unsolved problems concerning it. *H. B. Curry.*

Hailperin, Theodore. **Remarks on identity and description in first-order axiom systems.** J. Symbolic Logic 19, 14–20 (1954).

The remark on identity states: in a first-order axiom system (S) with a finite number of non-logical constants $[\Phi(x), \Psi(x, y), \cdots]$ the conjunction of the formulae:

$$\Phi(x)\equiv\Phi(y),\quad (z)[\Psi(x, z)\equiv\Psi(y, z).\&.\Psi(z, x)\equiv\Psi(z, y)],\ \cdots$$

satisfies the axiom schema for identity $(x=y)$, namely $x=x$, $x=y\&F(x).\supset F(y)$ where F is an arbitrary formula of the quantification theory of (S) with identity. The remarks on descriptions (ιxP: that x which has P) are intended to provide three methods of extending first-order axiom systems with identity which ensure that ιxP may be interpreted as follows: if ιxP is a proper description, i.e. $(Ey)(x)(x=y\equiv P)$, then ιxP is that x which has P; if ιxP is an improper description, in system (i) ιxP denotes nothing, i.e. $(y)(y\neq\iota xP)$, and every primitive predicate statement having ιxP as argument is false, in (ii) ιxP again denotes nothing, but now every primitive predicate statement having ιxP as argument is true, in (iii) ιxP denotes some one (arbitrarily chosen) thing, e.g. the unique x which satisfies some specific predicate Q_1. *G. Kreisel.*

Have

★**Götlind, Erik. A note on Chwistek and Hetper's foundation of formal metamathematics.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 268–270. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

Certain shortcomings of Chwistek and Hetper's system [J. Symbolic Logic **3**, 1–36 (1938)] are indicated. The question is raised as to whether a relation of identity is implicitly assumed. The handling of the concept of truth is criticized. The paper's major concern is with weaknesses which stem from the nominalistic point of view.

R. Barcan Marcus (Evanston, Ill.).

von Wright, G. H. On double quantification. Soc. Sci. Fenn. Comment. Phys.-Math. **16**, no. 3, 14 pp. (1952).

The author is here concerned with the general solution to the decision problem for a fragment of the lower functional calculus. In a previous paper [same Comment. **15**, no. 10 (1950); these Rev. **13**, 521] the solution was illustrated for a specific case. The fragment under consideration, called the theory of double quantification, includes only formulae involving two-place predicates and no free individual variables. In addition to solving the decision problem, the author also shows that all formulae of the theory of double quantification have a realization in a finite realm of individuals.

R. Barcan Marcus.

Rose, Alan. Self-dual primitives for modal logic. Math. Ann. **125** (1952), 284–286 (1953).

A given four-argument function, the constants n (denoting a necessary statement) and i (denoting an impossible statement) are taken as primitive for modal systems. The author shows that if $\sim\Diamond i$ and $\sim\Diamond\sim n$ are added to Lewis' axioms, then for S2 and stronger calculi, the three primitive connectives are independent and self-dual. The author mentions, in closing, a self-dual three-argument function proposed by B. L. van der Waerden to replace the given four-argument function.

R. Barcan Marcus.

Vredenduin, P. G. J. The logic of negationless mathematics. Compositio Math. **11**, 204–270 (1953).

The author tries to formalize the logic of Griss' negationless mathematics. Because Griss admits only true propositions as meaningful, a calculus of propositions cannot be constructed. In the first order predicate calculus only such predicates are meaningful which are satisfied by at least one element; thus, in the case of one variable, if a formula $p(x)$ occurs in a deduction, $(Ex)p(x)$ must be deducible. This circumstance causes considerable complications. Negation is not a primitive concept; it is supposed that in the domain of individuals there is defined an apartness relation $\#$, satisfying axioms analogous to those for the apartness relation in the domain of real numbers. Then a negation $\sim$ can be defined by $\sim p(x) =_d p(y) \rightarrow_y y \# x$. The logical constants of the second order functional calculus are distinguished from those of the first order calculus by a subscript 1. Second order negation is defined as follows:

$$\sim_1 p(x) =_d f(x) \rightarrow_f (Eu)(p(u) \wedge .f(v) \rightarrow_v u \# v) \vee (Eu)(p(v) \rightarrow_v u \# v. \wedge f(u)).$$

By means of the second order calculus the author proves that from $(Ex)\sim p(x)$ there can be deduced $p(x) \vee \sim p(x)$, a result which is not true in the intuitionistic interpretation. It has not yet been clarified at which point the system begins to diverge from the intuitionistic one. The author investigates the relations of his system with classical logic, with the intuitionistic logic and with Griss' outline of a logic, but the later work of Gilmore [Nederl. Akad. Wetensch. Proc. Ser. A. **56**, 162–174, 175–186 (1953); these Rev. **14**, 1053] could not be taken into account.

A. Heyting.

ALGEBRA

Bizley, M. T. L. Derivation of a new formula for the number of minimal lattice paths from (0, 0) to (km, kn) having just t contacts with the line $my=nx$ and having no points above this line; and a proof of Grossman's formula for the number of paths which may touch but do not rise above this line. J. Inst. Actuar. **80**, 55–62 (1954).

Write $\phi(k, t)$ for the lattice paths with t contacts as described in the title and, following Grossman [Scripta Math. **16**, 207–212 (1950); these Rev. **12**, 665], write

$$F_j=[j(m+n)]^{-1}\binom{jm+jn}{jm};$$

then the author's new formula may be stated as

$$\sum\phi(k, t)x^k=[1-\exp(-F_1x-F_2x^2-\cdots)]^t.$$

The corresponding formula for $\phi_k=\sum\phi(k, t)$, namely,

$$\sum\phi(k)x^k=\exp(F_1x+F_2x^2+\cdots)-1$$

is equivalent to Grossman's formula in the reference cited above. Various consequences of these generating functions are examined.

J. Riordan (New York, N. Y.).

Murty, V. N. Analysis of a triple rectangular lattice design. Biometrics **9**, 422–424 (1953).

A numerical illustration of the analysis of a triple rectangular lattice design is given following the methods of K. R. Nair [Biometrics **7**, 145–154 (1951); these Rev. **13**, 98], who pointed out that a triple rectangular lattice when $v=3\times4$ is a partially balanced incomplete block. The data are taken from p. 295 of "Experimental designs" [Wiley, New York, 1950; these Rev. **11**, 607] by Cochran and Cox.

T. Kitagawa (Fukuoka).

Roy, Purnendu Mohon. Rectangular lattices and orthogonal group divisible designs. Calcutta Statist. Assoc. Bull. **5**, 87–98 (1954).

An incomplete block design with $D=pq$ varieties each replicated r times in b blocks of size k each is called Orthogonal Group Divisible (OGD) if the pq varieties can be divided into two orthogonal sets of groups, one set containing p groups each with q varieties and the other set containing q groups each with p varieties, such that varieties belonging to the same group of the first set occur together λ_1 times, varieties belonging to the same group of the second set occur together λ_2 times and varieties not belonging to the same group of any set occur together λ_3 times. An OGD is a partially balanced incomplete block design (pbib). Its dual turns out to be a latinized rectangular lattice design. The author also studies relations between other classes of designs as, for instance, near balanced rectangular lattices and pbib.

H. B. Mann (Columbus, Ohio).

Popov, B. S. A note about the sums of binomial coefficients. Bull. Soc. Math. Phys. Macédoine 4 (1953), 5–6 (1954).

An alternative proof of the identity

$$\sum_{\lambda=0}(-2)^{-\lambda}\binom{n}{k+\lambda}\binom{n+k+\lambda}{\lambda}=\begin{cases}(-1)^{\nu}2^{-2\nu}\binom{n}{\nu} & \text{if } n-k=2\nu,\\ 0 & \text{otherwise,}\end{cases}$$

previously proven by the reviewer [Amer. Math. Monthly 60, 179–181 (1953); these Rev. 14, 642] and by Carlitz [ibid. 60, 181 (1953); these Rev. 14, 642].

E. Grosswald (Philadelphia, Pa.).

Duparc, H. J. A., and Peremans, W. A property of positive definite matrices. Math. Centrum Amsterdam. Rapport ZW 1954-006, 5 pp. (1954).

The mth compound of a positive definite matrix is positive definite. If $Z_1, \cdots, Z_k$ are $n\times m$ matrices, $m\leqq n$, and if A is a positive definite $n\times n$ matrix, then

$$\det_{1\leqq r,s\leqq k}[\det(Z_r^*AZ_s)]$$

is not negative. *J. L. Brenner.*

Vivier, Marcel. Note sur la structure des matrices unitaires. C. R. Acad. Sci. Paris 238, 1957–1959 (1954).

Various factorizations of a unitary matrix are obtained, one involving n^2 unitary matrices of one parameter together with permutation matrices. Thus the most general unitary matrix of order five is said to be expressible in the form

$$D_1M_{12}D_1M_{23}M_{12}T_2D_2M_{34}M_{23}M_{12}T_3D_3M_{45}M_{34}M_{23}M_{12}T_4D_4,$$

where $D_s=\|\operatorname{diag}(e^{i\theta_1}, \cdots, e^{i\theta_s}, 1, \cdots, 1)\|$, M_{ij} is a (real) rotation in the x_ix_j-coordinate plane, T_s is a permutation matrix acting on only the first s coordinates and the D's and M's contain a total of 15 and 10 independent parameters, respectively, the latter generalizing the Euler angles of real three space. [Reviewer's note: since the basic factorization requires complementary principal minors to be nonsingular (cf. eq. (1)), it would appear that the complete proof might require additional permutation matrix factors.]

W. Givens (New York, N. Y.).

Horn, Alfred. On the eigenvalues of a matrix with prescribed singular values. Proc. Amer. Math. Soc. 5, 4–7 (1954).

The author defines the singular values of a square matrix A over the complex field as the non-negative square roots of the eigenvalues of A^*A, where A^* is the transposed conjugate of A. A pair of n-tuples $(\lambda_1, \cdots, \lambda_n)$, $(\alpha_1, \cdots, \alpha_n)$ is said to be allowable if there exists an $n\times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$ and singular values $\alpha_1, \cdots, \alpha_n$. H. Weyl [Proc. Nat. Acad. Sci. U. S. A. 35, 408–411 (1949); these Rev. 11, 37] proved that if $[(\lambda_r), (\alpha_r)]$ is an allowable pair, arranged so that $\alpha_1\geqq\cdots\geqq\alpha_n\geqq 0$ and $|\lambda_1|\geqq\cdots\geqq|\lambda_n|$, then

$$\prod_{r=1}^{k}|\lambda_r|\leqq\prod_{r=1}^{k}\alpha_r \quad (1\leqq k\leqq n),$$

$$\prod_{r=1}^{n}|\lambda_r|=\prod_{r=1}^{n}\alpha_r.$$

The author proves the converse result that any pair $[(\lambda_r), (\alpha_r)]$ satisfying these conditions is allowable; moreover, there is a Hermitian matrix $H=[h_{\mu\nu}]$ with eigenvalues $\alpha_1, \cdots, \alpha_n$, such that

$$\begin{vmatrix}h_{11} & \cdots & h_{1k}\\ \cdot & \cdots & \cdot\\ h_{k1} & \cdots & h_{kk}\end{vmatrix}=|\lambda_1|\cdots|\lambda_k| \quad (1\leqq k\leqq n).$$

F. Smithies (Cambridge, England).

Krull, Wolfgang. Galoissche Theorie und Eliminationstheorie. Revista Acad. Ci. Madrid 47, 469–494 (1953). (Spanish summary)

It is well known that the task of determining the Galois group $\mathfrak{G}$ of a polynomial

$$P(x)=x^n-a_1x^{n-1}+a_2x^{n-2}-\cdots+(-1)^na_n$$

over a ground field K can be reduced, at least in principle, to decomposing a certain polynomial $R(w)$ in $n+1$ variables $w_0, w_1, \cdots, w_n$ into its irreducible factors. The Galois group is here taken to be the group of those permutations which may be applied to all algebraical relations (over K) between the roots of $P(x)$. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be these roots and let $\{\alpha_1^{(j)}, \cdots, \alpha_n^{(j)}\}$ $(j=0, 1, \cdots, n!-1;\ \alpha_i^{(0)}=\alpha_i)$ be the $n!$ n-tuples obtained by permuting them in all possible ways. Then

$$R(w)=\prod_{j=1}^{n!-1}(w_0+\alpha_1^{(j)}w_1+\cdots+\alpha_n^{(j)}w_n)$$

has coefficients in K. Every irreducible factor $I(w)$ of $R(w)$ leads to a determination of $\mathfrak{G}$ or one of its conjugates by the rule that the permutation $\alpha_i\to\alpha_{i'}$ belongs to the group if and only if the inverse permutation $w_{i'}\to w_i$ leaves $I(w)$ invariant.

The problem to which the present memoir is devoted may be briefly described as follows. Suppose that some nontrivial relations among the roots are known, i.e., that there are non-symmetric polynomials $p_\mu(u)\equiv p_\mu(u_1, \cdots, u_n)$ $(\mu=1, 2, \cdots, M)$ such that $p_\mu(\alpha)=0$. To what extent does this information facilitate the task of finding $\mathfrak{G}$?

The author begins by pointing out that the polynomial $R(w)$, which is used in the classical theory, may be interpreted as the w-resultant of the system

$$v_1+\cdots+v_n-a_1v_0=0,\ v_1v_2+\cdots+v_{n-1}v_n-a_2v_0^2=0,\ \cdots,\ v_1v_2\cdots v_n-a_nv_0^n=0,$$

together with a general linear form $w_0v_0+\cdots+w_nv_n=0$ in the indeterminates $w_0, w_1, \cdots, w_n$. When further homogeneous equations

$$v_0^{f_\mu}p_\mu(v_1/v_0, \cdots, v_n/v_0)=0,$$

where f_μ is the degree of p_μ, are added to the system, the elimination of the v's yields a resultant $E(w)$ which in general is a proper divisor of $R(w)$. It is now the irreducible factors of $E(w)$ that determine the group $\mathfrak{G}$. The lowering of the degree of the resultant may truly be regarded as a simplification of the problem. For, the calculation of the resultant is a mechanical process, however laborious, whilst its decomposition into irreducible factors is a matter of much greater difficulty and is feasible only in certain fields.

The solution sketched above applies to the simplest case in which the coefficients of $P(x)$ lie in K. The greater part of the paper is concerned with the more complex situation in which $\mathfrak{R}=K(a_1, \cdots, a_n)$ is an extension field of K, which is either algebraic or else of positive degree of transcendence over K. Results similar to those when $\mathfrak{R}=K$ are established by arguments based on deep methods of ideal theory, including the properties of the ground-polynomial developed by the author [Arch. Math. 1, 129–137 (1948); these Rev. 11, 310].

W. Ledermann (Manchester).

Abstract Algebra

Frink, Orrin. Ideals in partially ordered sets. Amer. Math. Monthly **61**, 223–234 (1954).

A set I of a partially ordered set P is called an order ideal of P if, whenever F is a finite subset of I, the set F^{*+} [G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 58; these Rev. **10**, 673] is also a subset of I. This definition has the following properties: 1) If $a \in P$, then the set of all $x \in P$ such that $x \leqq a$ is an order ideal; 2) the intersection of any number of order ideals is an order ideal; 3) in a lattice, order ideals are lattice ideals and conversely. The definition of order ideal is applied in rings and semigroups, which are quasiordered by a suitable relation. In this case the set of all left multiples of an element is a left order ideal and the collection of all left order ideals is a complete lattice. In the author's opinion the theory of order ideals will be useful in investigating various questions concerning the imbedding of a partially ordered set in a lattice and analogous problems of topology.

An order ideal is called completely irreducible if it is not the intersection of a collection of order ideals all distinct from it. A topology of a partially ordered set is called an ideal topology if the collection of all completely irreducible ideals and dual ideals is taken as a sub-basis for the open sets. This ideal topology gives the correct topology for chains, and also the correct topology for direct products of a finite number of ordered sets, namely, the Cartesian product topology. Some ways are suggested as to how the notion of an ordered algebraic system might be generalized.

M. Novotný (Brno).

Howson, A. G. Divisibility closure operations. J. London Math. Soc. **29**, 368–373 (1954).

In Proc. London Math. Soc. (3) **2**, 326–336 (1952) [these Rev. **14**, 238] G. Higman proved several combinatorial theorems, one of which asserts that if A is a set with a quasi-order (reflexive, transitive relation) then the set of finite sequences of elements of A with a naturally induced quasi-order satisfies the ascending chain condition (finite basis property) on dual ideals if A does. Every quasi-ordered set is a topological space when the closed sets are chosen to be the dual ideals. This suggests a generalization of Higman's theorem to the case of any topological space A with the ascending chain condition on closed sets replacing the ascending chain condition on dual ideals. The author proves this generalization in case the topology of A satisfies two extra hypotheses (not necessarily satisfied by every quasi-order topology).

D. Zelinsky (Evanston, Ill.).

Fujiwara, Tsuyoshi, and Murata, Kentaro. On the Jordan-Hölder-Schreier theorem. Proc. Japan Acad. **29**, 151–153 (1953).

It is shown that the Jordan-Hölder-Schreier Theorem holds for "modular elements" in any lattice. An element a is called modular if and only if

(i) $x \geqq a$ implies $(x \cap y) \cup a = x \cap (y \cup a)$

and

(ii) $x \geqq y$ implies $(x \cap a) \cup y = x \cap (a \cup y)$.

[For relevant literature and definitions see G. Birkhoff, "Lattice theory", Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 89; these Rev. **10**, 673.]

G. Birkhoff (Cambridge, Mass.).

Molinaro, Italico. Généralisation de l'équivalence d'Artin. II. C. R. Acad. Sci. Paris **238**, 1767–1769 (1954).

[For part I see same C. R. **238**, 1284–1286 (1954); these Rev. **15**, 675.] In a [residuated] gerbier (semi-lattice-ordered demi-group) T, if for a congruence R, T/R is a group with x as the maximal element of its class, then R is the Artin equivalence A_x, $a \equiv b$ (A_x) if and only if $x:a = x:b$. An element x is called nomal if $u(x:xu) \equiv e$ (A_x); a gerbier is called nomal if it contains a nomal element x, and then A_x is called the nomal equivalence. A nomal gerbier is called closed if $a:(x:x) \equiv a$ (A_x) for all a, or equivalently if $e:(x:x) \equiv e$ (A_x); properties thereof are studied. In particular, in a closed nomal gerbier, the nomal equivalence A_x is regular with respect to residuals; if the gerbier is also lattice-ordered, A_x is regular with respect to intersection.

P. M. Whitman (Silver Spring, Md.).

Aubert, Karl Egil. Généralisation de la théorie des r-idéaux de Prüfer-Lorenzen. C. R. Acad. Sci. Paris **238**, 2214–2216 (1954).

The author remarks upon the wide applicability, especially in the theory of rings and lattices, of the theory of an arbitrary r-system of ideals, introduced into fields by Prüfer [J. Reine Angew. Math. **168**, 1–36 (1932)] and into semi-groups by Lorenzen [Math. Z. **45**, 533–553 (1939); these Rev. **1**, 101]. He observes that the notion of an r-system of ideals can be defined in any commutative demi-group, and that, in the case of an r-system "of finite character", one can apply the methods used by Krull [Math. Ann. **101**, 729–744 (1929)] in the theory of ideals in commutative rings without chain condition. The author wishes to call attention to the following misprint. Axiom (3) should read: $\mathfrak{a}\mathfrak{b}_r \subseteq \mathfrak{b}_r \cap (\mathfrak{a}\mathfrak{b})_r$.

A. H. Clifford.

Szendrei, J. On rings admitting only direct extensions. Publ. Math. Debrecen **3** (1953), 180–182 (1954).

It is well known that a group G is a direct factor of every group containing it as a normal subgroup if and only if G is complete. The author remarks that the corresponding question for rings has an even simpler answer. A ring R is a direct summand of every ring containing it as a two-sided ideal if and only if R has a unit element. The sufficiency of the condition [see also Brown and McCoy, Proc. Amer. Math. Soc. **1**, 165–171 (1950); these Rev. **11**, 638] is due to the fact that the unit element of a ring is contained in the centre of an arbitrary extension ring. This holds equally for every element of the centre that is not a zero-divisor of the ring. Hence an integral domain is contained in the centre of every extension ring.

K. A. Hirsch (Boulder, Colo.).

Amitsur, A. S. Non-commutative cyclic fields. Duke Math. J. **21**, 87–105 (1954).

A division ring $\mathfrak{Z}$ is called a cyclic extension of a division ring $\mathfrak{F}$, if $\mathfrak{Z}$ is a right $\mathfrak{F}$-module of dimension n, and $\mathfrak{Z}$ possesses a cyclic group G of n automorphisms with $\mathfrak{F}$ as its fixed subring. As in the commutative case the author divides the study into the case where $n = p^e$, for p the characteristic of $\mathfrak{F}$, and the case where n is prime to the characteristic of $\mathfrak{F}$. The commutative extensions of degree and characteristic p are known to be obtained by the adjunction of a root of an irreducible polynomial $x^p - x - a$. The author shows every cyclic extension of degree p of $\mathfrak{F}$ is obtained as the difference ring $\mathfrak{F}[t] - \mathfrak{A}$ where $\mathfrak{F}[t] = \mathfrak{F}[t, D]$ is the ring of all differential polynomials relative to a derivation D of $\mathfrak{F}$, and $\mathfrak{A}$ is the two-sided ideal $(t^p - t - a)\mathfrak{F}[t]$. This construction parallels and generalizes the commutative case, and the

extension theory for extensions of degree $n=p^e$ and characteristic p, also parallels the commutative case. However, while every cyclic commutative field of degree p over $\mathfrak{F}$ can be imbedded in a cyclic commutative field of degree p^e over $\mathfrak{F}$ for e arbitrary, the conditions for existence of extensions in the noncommutative case are fairly complicated and appear to be necessary. The results for the case where n is prime to p also parallel the commutative case. The author also obtains some auxiliary results on extensions which are essentially cyclic extensions of the center of $\mathfrak{F}$, and on inner automorphisms of $\mathfrak{Z}$ over $\mathfrak{F}$. *A. A. Albert.*

Conrad, Paul. On ordered division rings. Proc. Amer. Math. Soc. **5**, 323–328 (1954).

Let R be a division ring with valuation f and with the additive value group Γ. All elements $r \in R$ with $f(r) \geqq 0$ form a subring $R^0(f)$ of R, called the valuation ring of f in R, and the residue class ring $R^0(f)/R_0(f)$ of $R^0(f)$ relative to the ideal $R_0(f)$ in $R^0(f)$ consisting of all elements $r \in R$ with $f(r)>0$ is called the residue class ring of R with respect to f. The main result of this paper states that a division ring R can be ordered if and only if there exists a valuation f of R for which the residue class ring $R^0(f)/R_0(f)$ of R with respect to f can be ordered so that every element of the form $R_0(f)+a_1^2\cdots a_n^2$ is positive, where $a_i \in R$ $(i=1, 2, \cdots, n)$ and $f(a_1^2\cdots a_n^2)=0$. This theorem generalizes to ordered division rings the relation between ordered fields and fields with valuations that was established by R. Baer [S.-B. Heidelberger Akad. Wiss. **1927**, no. 8, 3–13] and W. Krull [J. Reine Angew. Math. **167**, 160–196 (1932)]. The author establishes also a relation between the ordering of R and the natural valuation of R, the latter being defined in terms of the notion of Γ-valuation due to the author [Amer. J. Math. **75**, 1–29 (1953); these Rev. **14**, 842]. He also obtains an extension theory for ordered division rings in terms of the induced group extensions. *T. Szele* (Debrecen).

Northcott, D. G. On the notion of a form ideal. Quart. J. Math., Oxford Ser. (2) **4**, 221–229 (1953).

Two homogeneous ideals A and A' in polynomial rings $K[X_1, \cdots, X_r]$ and $K[X_1', \cdots, X_s']$ respectively are said to be H-equivalent if there exists an integer $p \geqq \max(r, s)$ and an isomorphism of $K[X_1, \cdots, X_r, X_{r+1}, \cdots, X_p]$ onto $K[X_1', \cdots, X_s', X'_{s+1}, \cdots, X_p']$ which maps forms into forms of the same degree and which transforms $(A, X_{r+1}, \cdots, X_p)$ onto $(A', X'_{s+1}, \cdots, X_p')$. The respective residue class rings of A and A' are then isomorphic. If Q is a local ring, M its maximal ideal, $K=Q/M$, and B an ideal in Q, the form ideal $\bar{B}$ of B relative to a given basis of M is defined à la Krull [J. Reine Angew. Math. **179**, 204–226 (1938)], except that nonminimal bases of M are allowed. $\bar{B}$ is a homogeneous polynomial ideal and is unique up to H-equivalence, wherefore it is meaningful to say that $\bar{B}$ is prime. If $\bar{B}$ is prime, so is B. [In a monograph of P. Samuel [Algèbre locale, Gauthier-Villars, Paris, 1953; these Rev. **14**, 1012] which appeared subsequent to the writing of the present article, a definition of $\bar{B}$ independent of any basis is given, and it is shown that primeness of $\bar{B}$ implies that of B (ibid., pp. 19–20).] *I. S. Cohen.*

Mikusiński, J. G.-. Sur la dérivée algébrique. Fund. Math. **40**, 99–105 (1953).

A est un anneau d'intégrité commutatif muni d'une dérivation (opération additive $x \to x'$ telle que $(ab)'=a'b+ab'$). L'anneau C des constantes est l'ensemble des éléments de dérivée nulle. Moyennant la condition supplémentaire (γ): "si $a'b-ab'=0$, a et b sont linéairement dépendantes", la condition de dépendance de n fonctions s'exprime par la nullité de leur Wronskien, et une équation différentielle linéaire sans second membre d'ordre n a au plus n solutions indépendantes. La dérivation s'étend au corps des fractions A_* de A par $(a/b)'=(a'b-ab')/b^2$. La condition (γ) est nécessaire et suffisante pour que le corps des constantes de A_* soit C_*. Application: A est l'algèbre de convolution des éléments $a\delta+f$, δ mesure de Dirac, f fonction continue de t nulle pour $t \leqq 0$; A_* est l'algèbre introduite par l'auteur dans des travaux antérieurs [Studia Math. **11**, 41–70 (1950); ces Rev. **12**, 189]; la dérivation est la multiplication par t; C est le corps des éléments $a\delta$ (multiples de l'unité δ); (γ) est vérifiée, donc $C_*=C$ est aussi le corps des constantes de A_*. *L. Schwartz* (Paris).

Lal, Goverdhan, Papy, Georges, et Sonnenschein, Jakob. Procédés de calcul en algèbre extérieure. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 238–246 (1954).

Let E be the exterior algebra on a vector space of dimension $2n$ over a field of characteristic 0; let $H \in E$ be homogeneous of degree 2, and regular (i.e. the nth exterior power H^n is not 0). An element $\rho_r \in E$, homogeneous of degree r, is called a residue modulo H if $H^{n-r+1}=0$ ($H^q=1$ whenever $q<0$). It is known that every homogeneous $\Omega_p \in E$ of degree p can be written uniquely in the form $\Omega_p=\sum_i \rho_{p-2i} \wedge H^i$, where ρ_{p-2i} is a residue modulo H of degree $p-2i$. The note under review gives a method for computing the coefficients ρ_{p-2i}. *E. R. Kolchin* (Paris).

Walker, Gordon L. Fermat's theorem for algebras. Pacific J. Math. **4**, 317–320 (1954).

Denote by $GF(p^n)[x]$ the domain of polynomials in the indeterminate x over $GF(p^n)$, and let A be the total matrix algebra of order m^2 over $GF(p^n)$. The author shows that the principal ideal of all polynomials $f(x)$ such that $f(a)=0$ for all $a \in A$ is generated by $f(m, p^n, x)$, the monic least common multiple of all polynomials of degree m in $GF(p^n)[x]$. He also proves that

$$f(m, p^n, x)=(x^{p^n}-x)(x^{p^{2n}}-x)\cdots(x^{p^{mn}}-x)$$

(extension of Fermat theorem). The results are extended to semi-simple algebras and to algebras with radical. *J. Levitzki* (Jerusalem).

Theory of Groups

Thierrin, Gabriel. Sur quelques classes de demi-groupes possédant certaines propriétés des semi-groupes. C. R. Acad. Sci. Paris **238**, 1765–1767 (1954).

A study is made of two properties of demi-groups generalizing that of cancellation. A demi-group is called limitative on the right if $ax=bx=a$ implies $a=b$. A demi-group is called completely "intègre" on the right if $xa=a^2$ implies $x=a$. Every demi-group which is completely intègre on both sides is limitative on both sides, but not conversely. If a demi-group D is limitative on the right (case 1), or completely intègre on the right (case 2), and if to each $a \in D$ there exists $a' \in D$ such that aa' and $a'a$ are idempotent, and moreover $aa'=a'a$ in case 2, then for every $a, b \in D$ the equation $xa=b$ has a unique solution $x \in D$. *A. H. Clifford* (Baltimore, Md.).

Thierrin, Gabriel. Sur la caractérisation des groupes par leurs équivalences régulières. C. R. Acad. Sci. Paris **238**, 1954–1956 (1954).

An equivalence relation R in a demi-group D is called "intègre" on the right if $xa \equiv a^2(R)$ implies $x \equiv a(R)$. R is called limitative on the right if $ax \equiv bx \equiv a(R)$ implies $a \equiv b(R)$. If R is intègre on both sides then it is limitative on both sides. Among the results shown, the following are typical. Let D be a demi-group in which every right regular equivalence relation R is right intègre, or else every such R is right limitative; then $a \in D$ implies $ax = a$ for some $x \in D$. A demi-group D is a group if and only if every right regular R is right intègre and also every left regular R is left intègre. A demi-group D with cancellation on both sides is a group if and only if every right regular R is right limitative.

A. H. Clifford (Baltimore, Md.).

Thierrin, Gabriel. Sur la caractérisation des groupes par leurs équivalences simplifiables. C. R. Acad. Sci. Paris **238**, 2046–2048 (1954).

P. Dubreil [Mém. Acad. Sci. Inst. France (2) **63**, no. 3 (1941); these Rev. **8**, 15] noted that every equivalence relation in a group is regular on the right if it is cancellable on the right. The author proves the following converse: If S is a demi-group in which the cancellation law holds on both sides, and if every right cancellable equivalence relation in S is right regular, then S is a group. *A. H. Clifford.*

Tamura, Takayuki. On finite one-idempotent semigroups. I. J. Gakugei, Tokushima Univ. (Nat. Sci.) **4**, 11–20 (1954).

The author studies finite semigroups having only one idempotent element. The Suschkewitsch kernel of such a semigroup S must be a group G, and $S^m = G$ for some positive integer m. The structure of S in the case $m=2$ is given completely. In the general case, we can arrange the elements in order $a_1, a_2, \cdots, a_n$ such that $G = \{a_1, \cdots, a_k\}$, and such that, setting $S_i = \{a_1, \cdots, a_i\}$ for $i = k+1, \cdots, n$, we have $SS_i \subseteq S_{i-1}$ and $S_iS \subseteq S_{i-1}$. If $k=1$ and S^2 has order $n-1$, then S is cyclic. *A. H. Clifford* (Baltimore, Md.).

Schwarz, Štefan. On maximal ideals in the theory of semigroups. I. Čehoslovack. Mat. Ž. **3**(78), 139–153 (1953). (Russian. English summary)

A proper left (right, two-sided) ideal of a semigroup S is called maximal if it is not contained in any other proper left (right, two-sided) ideal of S. This and the paper reviewed below (cited as II) are concerned with semigroups S containing a maximal left (right, two-sided) ideal $L^*(R^*, M^*)$ which is not only unique, but such that every other proper left (right, two-sided) ideal of S is contained in $L^*(R^*, M^*)$. The expression "$L^*(R^*, M^*)$ exists" is to be construed in this strong sense. Let L^* exist in the semigroup S. Assume that $a \in Sa$ for every $a \in S$. Then the complement $S - L^*$ of L^* in S is a subsemigroup of S. The same result is proved in II, replacing the condition "$a \in Sa$ for every $a \in S$" by "$S - L^*$ contains more than one element". If L^* happens to be a right as well as left ideal, then $S - L^*$ is left-simple. This is the case if every element of S has finite order. Let S be a semigroup satisfying the ascending chain condition for left ideals. If S contains at least one proper two-sided ideal, and if L^* exists, then so does M^*. The same result is proved in II without assuming the chain condition.

A. H. Clifford (Baltimore, Md.).

Schwarz, Štefan. On maximal ideals in the theory of semigroups. II. Čehoslovack. Mat. Ž. **3**(78), 365–383 (1953). (Russian. English summary)

This paper continues the foregoing. It is shown that every minimal left ideal of a semigroup with zero is the disjoint class sum of its radical and a left-simple subsemigroup. Corollary: If S contains no proper left ideal $\neq (0)$, then $S - (0)$ is a left-simple semigroup. If S is a simple semigroup with zero, containing at least one minimal left ideal l, and at most one maximal left ideal L, then $l = S$ and $L = (0)$. These and similar results are applied to the Rees quotient semigroup S/M, where M is a maximal two-sided ideal of S. For example: let S be a semigroup containing at least one proper two-sided ideal; let L^* exist (then so does M^*); let $S - L^*$ contain more than one element; let S/M^* contain a minimal left ideal; then $L^* = M^*$ and $S - L^*$ is a left-simple semigroup without zero. The paper concludes by noting that L^* always exists in a semigroup having a right identity element and at least one proper left ideal, and, in the above results, $S - L^*$ is a sum of disjoint isomorphic groups. *A. H. Clifford* (Baltimore, Md.).

Schwarz, Štefan. Contribution to the theory of torsion semigroups. Čehoslovack. Mat. Ž. **3**(78), 7–21 (1953). (Russian. English summary)

Let S be a torsion semigroup, i.e., one in which every element has finite order. Let e be an idempotent element of S. A subsemigroup P of S is called a maximal subsemigroup belonging to e if (1) $e \in P$, (2) P contains no idempotent element $\neq e$, and (3) no subsemigroup of S containing at most one idempotent element can properly contain P. Such a P always exists, but need not be unique. It is unique if S is commutative, or if S is totally non-commutative in the sense that distinct idempotent elements of S never commute. On the other hand, there is always a unique maximal subgroup of S containing e. *A. H. Clifford.*

Lyapin, E. S. Semigroups in all of whose representations the operators have fixed points. I. Mat. Sbornik N.S. **34**(76), 289–306 (1954). (Russian)

By a representation of a semigroup $\mathfrak{A}$ we mean a homomorphism of $\mathfrak{A}$ onto a semigroup of single-valued mappings ("operators") of some set Ω into itself. We write the operators to the left of the elements of Ω. Let P be the class of all semigroups $\mathfrak{A}$ such that, in every representation of $\mathfrak{A}$, every representing operator has a fixed point in Ω. Then $\mathfrak{A} \in P$ if and only if every element A of $\mathfrak{A}$ has a right zero in $\mathfrak{A}$, i.e. an element U of $\mathfrak{A}$ such that $AU = U$. Let Q and Q_0 be classes of semigroups. Q_0 is called a basic class for Q if (1) a semigroup belongs to Q if and only if it is a class sum of semigroups belonging to Q_0, and (2) Q_0 does not contain a proper subclass with the same property. For example, the class of all cyclic groups is a basic class for that of all semigroups admitting relative inverses. The main result of the paper is the determination of a basic class P_0 for P, as follows.

Let $\mathfrak{M}$ and $\mathfrak{B}$ be two disjoint semigroups, and let φ be a homomorphism $M \to \bar{M} = \varphi M$ of $\mathfrak{M}$ onto a semigroup $\bar{\mathfrak{M}}$. The "right-annihilating product $\mathfrak{B} \times_\varphi \mathfrak{M}$ of $\mathfrak{B}$ by $\mathfrak{M}$ with respect to φ" is defined to be the semigroup consisting of $\mathfrak{M}$ and of the set of all pairs $[B, \bar{M}]$ with $B \in \mathfrak{B}$, $\bar{M} \in \bar{\mathfrak{M}}$, products being defined as follows: $[B_1, \bar{M}_1]M_2 = [B_1, \overline{M_1M_2}]$, $M_2[B_1, \bar{M}_1] = [B_1, \bar{M}_1]$, $[B_1, \bar{M}_1] \cdot [B_2, \bar{M}_2] = [B_1B_2, \bar{M}_2]$. Next, $\mathfrak{T}_1$ ($\mathfrak{T}_2$) is defined to be the semigroup generated by an infinite sequence of symbols $T_1, T_2, \cdots$ subject to the

generating relations $T_iT_j=T_jT_i=T_j$ ($T_iT_j=T_j$) if $i<j$. A cyclic semigroup is called holoid if it is generated by an element A of finite index m and period 1 (i.e. $A^{m+1}=A^m$ with minimal m). P_0 consists of all holoid cyclic semigroups, and of all right annihilating products $\mathfrak{B}\times_\varphi\mathfrak{M}$, where $\mathfrak{M}$ is a non-holoid cyclic semigroup, $\mathfrak{B}$ is $\mathfrak{T}_1$ or $\mathfrak{T}_2$ or a one-element semigroup, and φ is either a homomorphism of $\mathfrak{M}$ onto a cyclic group or else an isomorphism of $\mathfrak{M}$ onto an infinite cyclic semigroup. *A. H. Clifford* (Baltimore, Md.).

Faragó, Tibor. Contribution to the definition of group. Publ. Math. Debrecen 3 (1953), 133–137 (1954).

Let G be a system of elements in which a single-valued binary multiplication is defined. Suppose G contains a unit element and that every element of G has a left inverse. If, in addition, G is associative, then G is a group. The author considers fifteen substitutes for the associative law, which are obtained by changing the position of the brackets and/or permuting the three elements a, b, and c in the right member of $(ab)c=a(bc)$. The associative law in G is replaced in turn by each of these fifteen substitutes. Nine of the resulting systems are commutative groups, three others are commutative systems not necessarily groups, while the remaining three (one of which is necessarily a quasigroup) need be neither commutative or associative. *D. C. Murdoch.*

Honda, Kin'ya. On primary groups. Comment. Math. Univ. St. Paul. 2 (1953), 71–83 (1954).

Let G be an (additive) abelian p-group and pG the subgroup of G consisting of all elements pg ($g\in G$). The author defines the horizontal exponent of a non-complete group G (i.e., provided that $pG\neq G$) as the smallest natural number n such that there exists an element of order p^n which is contained in G but not contained in pG. The main result of the paper is the following decomposition theorem: If G is of horizontal exponent n, then G can be represented as the direct sum $A+B$, where the subgroup A of G is a direct sum of cyclic groups of order p^n, and the subgroup B of G has a horizontal exponent greater than n or is complete. This result contains a great number of known structure theorems; among them the author points out the fundamental decomposition theorem for finite abelian groups, Kuroš's theorem on abelian groups with descending chain condition for subgroups [Math. Ann. 106, 107–113 (1932)], and Prüfer's theorem on abelian torsion groups of finite rank [Math. Z. 17, 35–61 (1923)].

The reviewer remarks that the author's theorem yields also a simple treatment of Kulikov's theory of basic subgroups [Mat. Sbornik N.S. 16(58), 129–162 (1945); these Rev. 8, 252] which is of fundamental importance in the structure theory of abelian p-groups of arbitrary power. *T. Szele* (Debrecen).

Itô, Noboru. On finite groups with given conjugate types. I. Nagoya Math. J. 6, 17–28 (1953).

Let G be a finite group and let $n_1>n_2>\cdots>n_r=1$ be all the numbers each of which is the index of the centralizer of some element of G in G. In this case G is said to be a group of type $(n_1, n_2, \cdots, n_r)$. With the present paper the author starts an investigation of the structure of groups of prescribed type. In this first paper he characterizes the groups of type $(n_1, 1)$. Any such group is nilpotent and $n_1=p^\alpha$ with a prime p. Moreover, it is shown that a group of type $(p^\alpha, 1)$ is the direct product of a p-group of the same type and of an abelian group. Therefore the study of groups of type $(n_1, 1)$ can be reduced to that of p-groups of type $(p^\alpha, 1)$. For such a group G the author proves the following theorem: G contains an abelian normal subgroup A such that any element ($\neq 1$) of the factor group G/A is of order p. In particular, if $p=2$, then G is metabelian. The number of elements of any generator system of a p-group G of type $(p^\alpha, 1)$, as well as the order of the subgroup of all the elements of order p in the center Z of G is not less than p^α. Moreover, if Z is cyclic, then $\alpha=1$ and G is of class 2. Also a number of typical examples are discussed.

In the remaining part of the paper the author gives some applications. A proper subgroup of G which is the centralizer of some element in G is called a fundamental subgroup of G. A fundamental subgroup F of G is said to be free if F contains no fundamental subgroup of G and is contained in no fundamental subgroup of G, properly. The author proves the following theorem: Let G be a finite non-simple centerless group every fundamental subgroup of which is free. Then G is metabelian with a factorization $G=AB$ where A is an abelian normal subgroup of G, B is a cyclic subgroup of G which coincides with its own normalizer, and the orders a and b of A and B respectively are relatively prime. The group G itself is of type $(a, b, 1)$. This generalizes a theorem of L. Weisner [Bull. Amer. Math. Soc. 31, 413–416 (1925)]. Also an improvement of a theorem of S. Čunihin [C. R. Acad. Sci. Paris 198, 531–532 (1934)] is obtained. *T. Szele* (Debrecen).

Itô, Noboru. Note on S-groups. Proc. Japan Acad. 29, 149–150 (1953).

Let G be a soluble group with maximal condition for subgroups. The author proves the following theorems. 1. If the Frattini subgroup $\Phi(G)$ contains the derived group of G, then G is nilpotent. [This has also been proved by the reviewer in Proc. London Math. Soc. (2) 49, 184–194 (1946); these Rev. 8, 132.] 2. The Frattini subgroup $\Phi(G)$ is nilpotent. [Another proof of this result by the reviewer is in the paper reviewed below; see also R. Baer, Math. Z. 59, 299–338 (1953); these Rev. 15, 598]. 3. If H is a subgroup of G whose derived group is a proper subgroup of the derived group of G, then the derived group of $\Phi(G)\cdot H$ is also a proper subgroup of the derived group of G. *K. A. Hirsch* (Boulder, Colo.).

Hirsch, K. A. On infinite soluble groups. V. J. London Math. Soc. 29, 250–251 (1954).

[For parts I–IV see Proc. London Math. Soc. (2) 44, 53–60, 336–344 (1938); 49, 184–194 (1946); J. London Math. Soc. 27, 81–85 (1952); these Rev. 8, 132; 13, 431.] In the paper reviewed above N. Itô has shown that the Frattini subgroup of a soluble group with maximum condition is nilpotent. The proof of this theorem which the author offers in this note is based on the lemma: If the Frattini subgroup ϕ of G is finitely generated, then a subgroup H of ϕ which is not normal in G possesses conjugates in G that are not conjugate in ϕ. It is worth noting that to assure validity of Itô's theorem it suffices to assume that the Frattini subgroup be a soluble group with maximum condition. [For further possibilities to generalize Itô's theorem, see R. Baer, Math. Z. 59, 299–338 (1953), in particular pp. 333–334; these Rev. 15, 598.] *R. Baer.*

Krull, Wolfgang. Über die Hauptreihen gewisser endlicher Gruppen. Acta Salmanticensia. Ciencias: Sec. Mat. no. 5, 13 pp. (1954).

R. Permutti has recently studied algebraic equations with supersoluble Galois group [Giorn. Mat. Battaglini (4)

4(80), 159–185 (1951); these Rev. 14, 8]. In the course of this work he established certain criteria for a finite group to be supersoluble. The present author generalizes and deepens these results in several directions. Amongst the number of interesting theorems which he proves, are the following: (1) A group of order $p^n r$, where p is a prime such that $p\equiv 1(r)$, is supersoluble if it has an invariant p-Sylow-group with Abelian factor group. (The case where r is a prime was discovered by Permutti.) (2) Let p, q be distinct primes such that $q\not\equiv 1(p)$ and let $s>1$ be the least integer such that $p^s\equiv 1(q)$. Every group of order $p^n q$, where $n<s$, is supersoluble, but if $n\geqq s$ there exist groups of order $p^n q$ which are not supersoluble. *W. Ledermann.*

Grün, Otto. Beiträge zur Gruppentheorie. V. Über endliche p-Gruppen. Osaka Math. J. 5, 117–146 (1953).

Parts I, II, III, IV have appeared earlier [J. Reine Angew. Math. 174, 1–14 (1935); 186, 165–169 (1945); Math. Nachr. 1, 1–24 (1948); 3, 77–94 (1949); these Rev. 10, 504; 12, 240] but the present part is independent of those preceding.

Regular p-groups have been introduced by P. Hall who developed also the theory of this important class of finite groups [Proc. London Math. Soc. (2) 36, 29–95 (1933)]. A finite p-group G is called regular if for any positive integer r and any pair of elements a, b in G there exist elements $c_1, \cdots, c_t$ belonging to the derived group of the subgroup $\{a, b\}$ of G generated by a, b such that $(ab)^{p^r}=a^{p^r}b^{p^r}c_1^{p^r}\cdots c_t^{p^r}$. P. Hall has shown that any finite p-group of class less than p is regular and consequently, in an arbitrary finite p-group G the subgroup $C_{p-1}(G)$ is regular, $C_{i-1}(G)$ denoting the ith member of the ascending central series of G. By virtue of the theory due to P. Hall several properties of abelian groups go over also to regular p-groups; thus, e.g., in a regular p-group all elements whose orders are $\leqq p^m$, resp., the p^mth powers of all elements, form a characteristic subgroup Ω_m, resp. $\mho_m$, for any positive integer m.

In the present paper the author proves a great number of theorems for arbitrary finite p-groups which in the special case of regular p-groups yield theorems of Hall. Illustrative of these are the following. Let G be in the sequel an arbitrary finite p-group of exponent p^k. If A is an abelian subgroup of G, then the subgroup generated by A and $C_{p-1}(G)$ in G is regular. Let $m\leqq k$ be an arbitrary natural number, and let $\Omega_m(G)$ denote a subgroup of G of exponent p^m which is maximal in the sense that it is no proper subgroup of a subgroup of G of exponent p^m. Moreover, let $\mho_m(G)$ denote a subgroup of G, maximal in the same sense, consisting only of elements which are p^mth powers in G. (If G is regular, then these subgroups are unique and coincide with the above groups Ω_m, $\mho_m$ respectively.) For a fixed m the intersection $D_m(G)$ [resp. $D_m^*(G)$] of all subgroups $\Omega_m(G)$ [$\mho_m(G)$] of G contains the uniquely defined subgroup $\Omega_m(C_{p-1}(G))$ [$\mho_m(C_{p-1}(G))$]. For fixed m, moreover, the intersection of the normalizers of all subgroups $\Omega_m(G)$ as well as that of all subgroups $\mho_m(G)$ contains the subgroup $C_p(G)$. The unique subgroup $\mho_m(C_{p-1}(G))$ lies in the centralizer of the subgroup $H_m(G)$ generated by all elements of order $\leqq p^m$ in G. Every normal subgroup with more than one element of G contains a regular normal subgroup with more than one element of G.

Let $P_m(G)$ denote the subgroup generated by all p^mth powers in G and, moreover, let $U_m^*(G)$, resp. $U_m(G)$, denote the set of all elements a of G such that for every $x\in G$ the relation $(ax)^{p^m}=a^{p^m}x^{p^m}$, resp. $(ax)^{p^m}=a^{p^m}$, holds. Then, for every positive integer m, $U_m^*(G)$ and $U_m(G)$ are characteristic subgroups of G and $U_m^*(G)\supseteq U_m(G)\supseteq\Omega_m(C_{p-1}(G))$. All elements of order $\leqq p^m$ in $U_m^*(G)$ form just the subgroup $U_m(G)$. The centralizer of the subgroup $U_m^*(G)$ contains $P_m(G)$. As an example for a dualism occurring several times throughout the paper we mention the following theorem of the author: Any element of order $\leqq p^m$ in $C_p(G)$ commutes with all p^mth powers in G, and any p^mth power in $C_p(G)$ commutes with all elements of order $\leqq p^m$ in G. The paper concludes with an investigation of the relation between the iterated series $P_m(G)$, $P_m(P_m(G))$, $\cdots$ and the ascending central series of G. *T. Szele* (Debrecen).

Grün, Otto. Über das direkte Produkt regulärer p-Gruppen. Arch. Math. 5, 241–243 (1954).

A p-group G is regular if, for any given $P, Q\in G$ and any positive integer α, there exist elements $S_1, S_2, \cdots$ in $[U, U]$, where $U=\{P, Q\}$, such that

$$(PQ)^{p^\alpha}=P^{p^\alpha}Q^{p^\alpha}\prod_\nu S_\nu^{p^\alpha}.$$

If G_1 and G_2 are regular p-groups, it is known (H. Wielandt, unpublished) that their direct product $G_1\times G_2$ need not be regular. The author proves that a sufficient condition for the regularity of $G_1\times G_2$ is that the commutator group of one of the factors, say G_2, have exponent p or 1. *P. Hall.*

Ōno, Katsuhiro, and Tsuboi, Teruo. Remarks on groups. Sci. Rep. Yokohama Nat. Univ. Sect. I. 2, 13–15 (1953).

Let G be a group, K its commutator subgroup and Z its center. It is shown that if G contains a normal abelian subgroup A with finite cyclic factor group G/A, then $A/(Z\cap A)\simeq K$. This generalizes a theorem of Tuan on p-groups [Acad. Sinica Science Record 3, 17–23 (1950); these Rev. 12, 800]. *R. Brauer* (Cambridge, Mass.).

Kochendörffer, Rudolf. Zur Theorie der Rédeischen schiefen Produkte. J. Reine Angew. Math. 192, 96–101 (1953).

L. Rédei [same J. 188, 201–227 (1950); these Rev. 14, 13] has introduced the "skew" product $G\circ\Gamma$ of two groups G and Γ with the composition rule: $(a, \alpha)(b, \beta)=(ab^\alpha\beta^a, a^b\alpha^b\beta)$, where $a, b\in G$, $\alpha, \beta\in\Gamma$. The skew product $G\circ\Gamma$ is called k-fold degenerate if precisely k of the following four relations hold identically in $G\circ\Gamma$: $b^\alpha=b$, $\beta^a=e$, $a^b=\epsilon$, $\alpha^b=\alpha$, where e, ϵ are the unit elements of G, Γ, respectively. It turned out [loc. cit.] that there are only four essentially distinct two-fold degenerate skew products. Two of these were studied by Rédei and coincide with the Schreier extensions of Γ by G and the Zappa product of G and Γ, respectively. In the present paper the author determines the structure of the remaining two, denoted by $G\circ_3\Gamma$ and $G\circ_4\Gamma$, respectively.

In $G\circ_3\Gamma$ the composition rule is $(a, \alpha)(b, \beta)=(ab\alpha^\beta, a^b\alpha\beta)$. It is assumed that (e, ϵ) is the unit element of $G\circ_3\Gamma$. Let G_1 be the subgroup of G generated by all elements α^β, Γ_1 the subgroup of Γ generated by all elements a^b. Then (G_1, Γ_1) is contained in the centre of $G\circ_3\Gamma$ and is isomorphic to the direct product $G_1\times\Gamma_1$. The group $G\circ_3\Gamma$ can be regarded as an extension of (G_1, Γ_1) by $(G/G_1)\times(\Gamma/\Gamma_1)$. The automorphism group induced in (G_1, Γ_1) is the identity, and the corresponding factor system can be described explicitly in terms of coset representatives of G_1 in G and Γ_1 in Γ. Also (G, Γ_1) is a normal subgroup of $G\circ_3\Gamma$ and can be regarded as extension of Γ_1 by G belonging to the factor system a^b and to the identity automorphism group. And $G\circ_3\Gamma$, in turn, is an extension of (G, Γ_1) by Γ/Γ_1 with identity automorphism group and an explicitly described factor system. A dual result holds for the normal subgroup (G_1, Γ) of G.

In $G \wr \Gamma$ the composition rule is: $(a, \alpha)(b, \beta) = (ab^{\alpha}, a^{b}\alpha\beta)$. It is again assumed that (e, ϵ) is the unit element of $G \wr \Gamma$. Then (G, Γ_1) is a normal subgroup of $G \wr \Gamma$ and is the extension of Γ_1 by G belonging to the factor system a^b and to the identity automorphism group. Finally $G \wr \Gamma$ is the extension of (G, Γ_1) by Γ/Γ_1 whose factor system and group of automorphisms are described explicitly. *K. A. Hirsch.*

Rédei, L., und Stöhr, A. Über ein spezielles schiefes Produkt in der Gruppentheorie. Acta Sci. Math. Szeged 15, 7–11 (1953).

Among the "skew products" $G \circ \Gamma$ of two groups G, Γ [see Rédei, J. Reine Angew. Math. 188, 201–227 (1950); these Rev. 14, 13] the class of Schreier extensions of Γ by G, written as $G \wr \Gamma$, and the factorizable (Zappa) extensions $G \wr \Gamma$ in which the multiplication of pairs is defined by $(a, \alpha)(b, \beta) = (ab^{\alpha}, \alpha^{b}\beta)$. From these two types all other groups of type $G \circ \Gamma$ can be obtained by a simple construction, as Kochendörffer has proved in the paper reviewed above. In the present note the authors show that groups with the multiplication rule $(a, \alpha)(b, \beta) = (a^{\beta}b, \alpha^{b}\beta)$ do not lead to any new type, but only represent special cases of $G \wr \Gamma$. *K. A. Hirsch* (Boulder, Colo.).

Holyoke, T. C. Transitive extensions of dihedral groups. Math. Z. 60, 79–80 (1954).

The permutation group H is a transitive extension of a subgroup G if G is the largest subgroup fixing one symbol. Using conditions proved in an earlier paper [Amer. J. Math. 74, 787–796 (1952); these Rev. 14, 616] the author proves that the only dihedral groups (including that containing an infinite cyclic group) which possess transitive extensions when represented as groups of permutations of the cosets of a reflection subgroup are those of orders four, six, and ten. *J. S. Frame* (East Lansing, Mich.).

Ibrahim, E. M. On a theorem by Murnaghan. Proc. Nat. Acad. Sci. U. S. A. 40, 306–309 (1954).

Murnaghan has mentioned that the irreducible characters of the $2k$-dimensional orthogonal group, denoted by $[\lambda]$, are connected with the irreducible characters $\langle\lambda\rangle$ of the $2k$-dimensional symplectic group by the relation

$$\langle\lambda\rangle = [\lambda^*]^*,$$

where * denotes conjugate partition. Thence, if (λ) is a partition of m and m is even,

$$\langle\lambda\rangle \otimes \{\mu\} = ([\lambda^*] \otimes \{\mu\})^*,$$

but if m is odd,

$$\langle\lambda\rangle \otimes \{\mu\} = ([\lambda^*] \otimes \{\mu^*\})^*.$$

The author gives a proof depending on generating functions and the theorem of conjugates due to the reviewer. A direct relation is also obtained between symplectic and orthogonal group characters, as follows:

$$[\lambda] = \langle\lambda\rangle + \sum \Gamma_{\eta\mu\lambda}\langle\mu\rangle,$$

where $\Gamma_{\eta\mu\lambda}$ is the coefficient of $\{\lambda\}$ in the product $\{\eta\}\{\mu\}$, and the summation is with respect to all partitions $\{\eta\}$ which appear in the expansion

$$\begin{aligned}\sum\{\eta\} = (1 - \{2\} + \{31\} - \{41^2\} - \{3^2\} - \{4^2 2\} + \{431\} + \cdots) \\ \times (1 + \{1^2\} + \{2^2\} + \{1^4\} + \cdots) \\ = 1 - \{2\} + \{1^2\} + \{2^2\} - \{21^2\} - \{2^3\} + \cdots.\end{aligned}$$

There is a similar relation expressing $\langle\lambda\rangle$ in terms of $[\lambda]$.

D. E. Littlewood (Bangor).

Honda, Kin'ya. Analytic considerations on finite groups and their representations. Comment. Math. Univ. St. Paul. 2 (1953), 41–46 (1954).

Let G be a finite group with h classes of conjugate elements. Then there exist exactly h non-equivalent irreducible representations D_i of G in the field of complex numbers $(i = 1, 2, \cdots, h)$. Let D be an arbitrary isomorphic representation of G and let $D^{(1)} = D$, $D^{(n+1)} = D^{(n)} \times D$ $(n = 1, 2, 3, \cdots)$ where $\times$ denotes Kronecker's product representation. Using methods of the theory of functions, the author proves the following theorems. If D is of degree d and if $D^{(n)}$ includes exactly u_{ni} irreducible representations equivalent to D_i, then for each fixed i, $\limsup (u_{ni})^{1/n} = d$ holds. Every irreducible representation of G can be obtained by decomposing one of the h representations $D^{(n)}$ $(n = 1, 2, \cdots, h)$. If M is a normal subset of G containing m elements, i.e., $g^{-1}Mg = M$ holds for any $g \in G$, and if G is generated by M, then for the number $v_n(a)$ of the ordered systems $b_1, \cdots, b_n$ $(b_j \in M;\ j = 1, \cdots, n)$ such that $b_1 \cdots b_n = a$, $\limsup (v_n(a))^{1/n} = m$ holds, a being an arbitrary fixed element of G. If G is generated by the normal subset M, then every element a of G can be written in the form $a = b_1 \cdots b_k$ $(b_j \in M;\ j = 1, \cdots, k)$ where $1 \leq k \leq h$ and k depends on the choice of a. *T. Szele* (Debrecen).

Takahashi, Shuichi. Dimension of compact groups and their representations. Tôhoku Math. J. (2) 5, 178–184 (1953).

Let G be a compact abelian group and G^* its dual. It is known that $\dim G = \operatorname{rank} G^*$. The author proves corresponding theorems for compact groups G not assumed to be abelian: (A) $\dim G$ equals the degree of transcendency over C (the complex numbers) of the representative ring of G over C; (B) $\dim \bar{G}$ (where $\bar{G}$ is the space of conjugate classes of G) equals the degree of transcendency over C of the algebra generated by the characters of the finite dimensional representations of G over C. *P. A. Smith.*

Jennings, S. A. Substitution groups of formal power series. Canadian J. Math. 6, 325–340 (1954).

Let R_0 be a commutative ring and R the ring of formal power series of x with coefficients in R_0. A power series of the form $f(x) = x + a_2x^2 + \cdots$, $a_i \in R_0$, defines in an obvious way an R_0-automorphism of R, and the author studies the structure of the group G formed by all such automorphisms of the ring R. Let G_r $(r \geq 1)$ denote the subgroup of G consisting of all automorphisms of the form $f(x) = x + a_{r+1}x^{r+1} + \cdots$. Then the sequence of groups $G = G_1 \supset G_2 \supset G_3 \supset \cdots$ gives a descending central chain of G with the intersection e and G can be made a topological group by taking those G_r as neighborhoods of e. Properties of the groups G_r are studied for various types of the ring R_0. In particular, if R_0 is a field of characteristic 0, $\{G_r\}$ coincides with the lower central series of G and, by means of the derivations of the ring R, one-parameter subgroups of G are defined as usual, making it possible to introduce canonical parameters in G. It is also proved in this case that G is topologically generated by two suitably chosen one-parameter subgroups.

K. Iwasawa (Cambridge, Mass.).

Mostow, G. D. Factor spaces of solvable groups. Ann. of Math. (2) 60, 1–27 (1954).

By a solvmanifold (nilmanifold) the author means a topological space M on which a solvable (nilpotent) Lie group of transformations G acts transitively, i.e. the factor space G/S of G modulo a closed subgroup S. The author

studies the topological structure of such $M=G/S$, and the main results are stated as follows. (1) Two compact solvmanifolds having the same fundamental group are isomorphic (a generalization of a theorem of Malcev proved for nilmanifolds). (2) A compact solvmanifold is a bundle over a toroid with a nilmanifold as fibre. (3) Any solvmanifold is regularly covered by the direct product of a compact solvmanifold and a euclidean space, the covering group being finite abelian. (4) If S is an "algebraically connected" subgroup of G, then $M=G/S$ is homeomorphic to the direct product of a compact solvmanifold and a euclidean space (algebraic connectivity is a notion which generalizes the usual topological connectivity and is defined by means of algebraic Lie groups).

The proof needs delicate analysis of both algebraic and topological structure of Lie groups and the key lemmas are as follows. (i) Let S be a closed subgroup of a solvable Lie group G with compact G/S and let N be the maximal connected nilpotent normal subgroup of G. If S includes no proper connected normal subgroup of G, then SN is closed. Moreover, the connected component of 1 in S is contained in N. (ii) Let E_i $(i=1, 2)$ be euclidean spaces and let Γ_i $(i=1, 2)$ be topological groups operating freely on E_i (i.e. $g(p)=g'(p)$ implies $g=g'$) so that each E_i is a fibre bundle with the orbits of Γ_i as fibres and that E_i/Γ_i is a toroid. If, then, there is an isomorphism θ of Γ_1 onto Γ_2 as topological groups, there exists a homeomorphism of E_1 onto E_2 which is equivariant with respect to θ. *K. Iwasawa.*

Wallace, A. D. A note on mobs. II. Anais Acad. Brasil. Ci. 25, 335–336 (1953).

This note is an abbreviated version of an earlier paper, lost in transit, which had been rendered redundant in large part by a paper of Numakura [Math. J. Okayama Univ. 1, 99–108 (1952); these Rev. 14, 18]. No proofs are given.

A mob S is a topological (Hausdorff) semigroup. Every algebraic subgroup of S is contained in a maximal algebraic subgroup M of S. Distinct maximal algebraic subgroups are disjoint. If S is compact (as will be assumed henceforth), M is a compact topological group. The set E of idempotents of S is non-empty [Numakura, loc. cit., and Wallace, Anais. Acad. Brasil. Ci. 24, 329–334 (1952); these Rev. 14, 724]. Let $H(e)$ denote the smallest algebraic subgroup of S containing $e \in E$, let $H=\bigcup\{H(e): e\in E\}$. Then H is closed and distinct $H(e)$'s are disjoint. If $x \in H$, let $\eta(x)=e$, and let $\theta(x)$ be the inverse of x in $H(e)$. Then $\eta: H\rightarrow E$ is a retraction and $\theta: H\rightarrow H$ is an involutory homeomorphism. Finally, if K is the minimal closed ideal of S, then

$$K=\bigcup\{H(e): e\in E\cap K\}=\bigcup\{eSe: e\in E\cap K\}.$$

M. Henriksen (Lafayette, Ind.).

NUMBER THEORY

Batty, J. S. Some properties of pure recurring decimals. Math. Gaz. 38, 90–95 (1954).

The author investigates some elementary properties of pure recurring decimals. *P. Erdös.*

Sierpiński, W. Remarques sur les progressions arithmétiques. Colloquium Math. 3, 44–49 (1954).

The author makes comments and raises questions about the existence of arithmetic progressions (a.p.) in various sequences. The first type of sequence is the geometric progression. It is easy to exhibit geometric progressions having three terms in a.p. To avoid the trivial a.p. of difference zero, one must take for geometric progression powers of an irrational root of a trinomial equation $x^n-2x^m+1=0$. The problem of four terms in a.p. appears to be unsolved. Among the squares three but not four terms of an a.p. may be found. The author asks whether three cubes can be in a.p. This was answered in the negative by Euler in 1770. Arithmetic progressions among square-roots and factorials are also discussed. Whether there are n terms in a.p. among the primes is proposed as a problem. Thébault's example of $n=9$ beginning with $p=199$ and ending with 2089 is cited.

Two theorems are proved about infinite a.p. The first of these asserts that if C is any given infinite ordered set of natural numbers then the set of all natural numbers may be partitioned into two disjoint sets A and B such that the terms of A exceed the corresponding terms of C and still B contains no infinite a.p. whatever. The second asserts that a sequence of natural numbers can be constructed which tends to infinity faster than a prescribed sequence and yet contains a.p. of arbitrary finite length. The author concludes with a theorem of Zarankiewicz: Let A be the class of all natural numbers n for which $n+[\sqrt{n}]$ is even. Let B be the complementary class; then neither A nor B contain a single infinite a.p. This is in contrast to a well known theorem of van der Waerden on finite a.p.'s.

D. H. Lehmer (Berkeley, Calif.).

Kanold, Hans-Joachim. Einige neuere Bedingungen für die Existenz ungerader vollkommener Zahlen. J. Reine Angew. Math. 192, 24–34 (1953).

In continuation of his former papers [see these Rev. 3, 268; 5, 33; 6, 255; 9, 78; 13, 443], the author obtains some new criteria for the existence of odd perfect numbers. The following results may be mentioned. There do not exist odd perfect numbers of the forms

$$p^a q_1^4 q_2^4 \prod_{\rho=3}^{r} q_\rho^2 \quad \text{and} \quad p^b \prod_{\rho=1}^{r} q_\rho^2 \prod_{\sigma=r+1}^{s} q_\sigma^4.$$

A. Brauer (Chapel Hill, N. C.).

Nagell, Trygve. On a special class of Diophantine equations of the second degree. Ark. Mat. 3, 51–65 (1954).

The author proves the following results using the elementary theory of quadratic fields but without using ideals. (I) Let $D>1$ be squarefree and consider equations of the type $Ax^2-By^2=E$ where $AB=D$ and $E=\pm1, \pm2$ with the supplementary conditions $1<A<B$ for $E=+1$, $1\leqq A<B$ otherwise. Further, $E\neq\pm2$ if D is even. Then precisely one of these equations is soluble. If $x=\xi$, $y=\eta$ is the least solution, then $|E|^{-1}(\xi\sqrt{A}+\eta\sqrt{B})^2=U+V\sqrt{D}$ is a fundamental solution of Pell's equation $U^2-DV^2=4$. (II) Suppose that Pell's equation $U^2-DV^2=4$ admits a solution with odd U, V. Then for $1\leqq A<B$, $AB=D$, precisely one of the equations $Ax^2-By^2=\pm4$ other than Pell's equation is soluble in odd integers x, y. If ξ, η is the least solution of this other equation then $\frac{1}{4}(\xi\sqrt{A}+\eta\sqrt{B})^2=U+V\sqrt{D}$ is a fundamental solution of Pell's equation. It is remarked that these results follow readily from known results of ideal theory; and that the author has already proved something rather analogous for cubic fields [J. Math. Pures Appl. (9) 4, 209–270 (1925)]. *J. W. S. Cassels.*

Nagell, Trygve. Verallgemeinerung eines Fermatschen Satzes. Arch. Math. 5, 153–159 (1954).

The Diophantine equation $x^2+2=y^n$ has no solutions for $n>1$ except $x=\pm 5$, $y=3$ in case $n=3$. This is proved with the aid of a theorem of K. Mahler [Arch. Math. Naturvid. 41, no. 6 (1935)] on the Pell equation. It is also established that for $n>2$, the equation $x^2+8=y^n$ has no solutions except perhaps when n is a prime of the form $8k\pm 1$, and for any such value of n there is at most one solution. *I. Niven.*

Đerasimović, B. Eine Diophantische Gleichung vom dritten Grade. Bull. Soc. Math. Phys. Serbie 5, no. 3–4, 61–77 (1953). (Serbo-Croatian. German summary)

The author generalizes that part of the A. Markov theory of minima of indefinite binary quadratic forms which relates the so-called Markov numbers to the diophantine cubic

$$x^2+y^2+z^2=3xyz. \tag{1}$$

The author considers the cubic $x^2+\epsilon_1 y^2+\epsilon_2 z^2=\alpha xyz$, where $\epsilon_1^2=\epsilon_2^2=1$. Solutions are given in terms of cumulants quite similar to those which generate the Markov numbers. The special role played by the number 2 in the case of (1) is now played by $\alpha-1$. [For the Markov theory see L. E. Dickson, Studies in the theory of numbers, Univ. of Chicago Press, 1930, Ch. 7.] *D. H. Lehmer.*

Grey, L. D. A note on Fermat's last theorem. Math. Mag. 27, 274–277 (1954).

The author considers the equation $x^{2p}+y^{2p}=z^{2p}$ in the case where p does not divide xyz. It is shown that p must be of the form $3n+1$ and that all odd prime factors of $p+1$ must be of the form $4m+1$. *D. H. Lehmer.*

Kanellos, S. G. On the Bernoulli's numbers. Bull. Soc. Math. Grèce 28, 101–106 (1954). (Greek. English summary)

The author's procedure for computing the Bernoulli numbers is the following: Let

$$\varphi_1(x)=1+(e^x-1)(x-2)/(2x)=\tfrac{1}{2}\sum_1^\infty nx^{n+1}/(n+2)!,$$

and let $\{\varphi_1(x)\}^k=\sum_1^\infty a_{n,k}x^{2k+n-1}$. Arrange these power series in a triangular array, so that $\{\varphi_1(x)\}^k$ forms the kth row and so that the nth column contains the $(n+1)$th powers of x. The resulting triangular matrix of coefficients can be readily computed since the kth row is simply the kth power (Cauchy product) of the first row. Now let κ_{n+1} denote the sum of the elements of the nth column of this matrix. The sequence $\{\kappa_n\}$ can then be used to compute the Bernoulli numbers by means of the recursion formulas: $\kappa_2=-B_2/2!$, $\frac{1}{2}\kappa_{p-1}-\kappa_p=-B_p/p!$ $(p\geqq 3)$. *T. M. Apostol.*

Zech, Th. Potenzsummen und Bernoullische Zahlen. Z. Angew. Math. Mech. 34, 119–120 (1954).

Let $S_k(n)=0^k+1^k+\cdots+(n-1)^k$ for $k\geqq 0$. After remarking that the familiar formula

$$S_k(n)=\frac{B_{k+1}(n)-B_{k+1}}{k+1} \tag{*}$$

is not intuitively evident, the author suggests that the symbolic formula $S_k(n+k)=S_k(n)+(S(x)+n)^k$ leads rapidly to (*). Alternatively since $\Delta S_k'(n)=kn^{k-1}$, it follows that $S_k'(n)-kS_{k-1}(n)=B_k$, whence $S_k'(x)=(B+n)^k$; also for $n=1$, we get $(B+1)^k=B^k$, $k\geqq 1$, so that the B_k are indeed the Bernoulli numbers. *L. Carlitz* (Durham, N. C.).

Vinogradov, I. M. An elementary proof of a theorem from the theory of prime numbers. Izvestiya Akad. Nauk SSSR. Ser. Mat. 17, 3–12 (1953). (Russian)

The following result is proved. Let $0\leqq\sigma\leqq 1$, let $\pi_\sigma(N,q)$ be the number of primes p not exceeding N such that $p/q-[p/q]<\sigma$; then $\pi_\sigma(N,q)=\sigma\pi(N)+O(R)$ where

$$R=N^{1+\epsilon}\{(1/q+q/N)^{1/2}+N^{-1/6}\}$$

so that for large q and N/q, $\pi_\sigma(N,q)$ is asymptotically $\sigma\pi(N)$. The author remarks that a modification of the method enables him to replace the exponent $-1/6$ of N by $-1/5$ and that the method is applicable to a number of other problems including the corresponding question for primes in a given arithmetic progression.

The method of proof is different from that in Chapter XI of the author's book [Trudy Mat. Inst. Steklov. 23 (1947); these Rev. 10, 599] which depended on the estimation of exponential sums; the final result in the book corresponds to the estimate with $-1/6$ replaced by $-1/5$ and has a few other refinements as well. The present proof depends on the use of the function $\psi_\sigma(x)$ defined to be $1-\sigma$ if $x-[x]<\sigma$ and to be $-\sigma$ otherwise. By applying devices similar to those the author previously used for giving elementary estimates for sums of the form $\sum_x(f(x)-[f(x)])$, he now gives upper bounds for the absolute values of sums of the form

$$\sum_x\psi_\sigma(\alpha x+\beta),\quad \sum_x\sum_y\psi_\sigma(xy/q+h/q),\quad \sum_x\sum_y\psi_\sigma(axy/q),$$
$$\sum_x\sum_y\sum_m\psi_\sigma(axym/q),\quad \sum_p\psi_\sigma(ap/q).$$

From the estimate of the last sum, the result immediately follows. The methods used here are technically simpler than those used in the author's book. *L. Schoenfeld.*

Vinogradov, I. M. Improvement of an estimate for the sum of the values $\chi(p+k)$. Izvestiya Akad. Nauk SSSR. Ser. Mat. 17, 285–290 (1953). (Russian)

The author improves his previous estimate [same Izvestiya 16, 197–210 (1952); these Rev. 14, 22] for the sum $\sum_{p\leqq N}\chi(p+k)$ where χ is a nonprincipal character modulo q and p is prime. If $cq^{3/4}\leqq N\leqq c'q^{3/4}$, then the sum has an estimate of the form

$$N^{1+\epsilon}(q^{1/4}N^{-1/3}+N^{-1/10})$$

in place of the earlier inferior estimate of $N^{1+\epsilon}q^{3/16}N^{-1/4}$. The method now used is very similar to that used in the earlier paper and still depends on Weil's estimate of the Kloosterman sum. However, a more elaborate argument enables the author to obtain the sharper estimate. *L. Schoenfeld.*

Klimov, A. I. On an estimate of a bound for the zeros of L-functions. Doklady Akad. Nauk SSSR (N.S.) 89, 205–208 (1953). (Russian)

Klimov, A. I. Improved estimate of a bound for the zeros of L-functions. Ukrain. Mat. Žurnal 5, 171–184 (1953). (Russian)

The first paper is a summary of the second. The main result is that for a character χ modulo k, the Dirichlet series $L(s,\chi)$ has no zeros in the region $|t|\geqq 1$,

$$\sigma\geqq 1-c/\{\log(k+3)+(\log\gamma\log\log\gamma)^{3/4}\},$$

where $\gamma=|t|+8$ and $c=.0005$. However, (for unspecified c) this was proved earlier by Tatuzawa [Jap. J. Math. 21, 93–111 (1952); these Rev. 15, 202]. The details are by now familiar, depending as they do on exponential sums. *L. Schoenfeld* (Princeton, N. J.).

Prachar, K. Über ein Problem vom Waring-Goldbach'schen Typ. Monatsh. Math. 57, 66–74 (1953).
Prachar, K. Über ein Problem vom Waring-Goldbach'schen Typ. II. Monatsh. Math. 57, 113–116 (1953).

In the first paper, the author proves that almost all (in the sense of asymptotic density 1) even numbers can be represented in the form $n=p_2^2+p_3^3+p_4^4+p_5^5$ where the p_r are primes. The proof makes use of a Farey dissection of the unit interval, the asymptotic estimate for the number of primes in a progression, Vinogradov's estimate of exponential sums involving prime summation variable, a number of estimates of Hua, and a result of Cauchy-Davenport-Chowla on the number of residue classes in the sum of two systems of residue classes.

In the second paper, the author shows that all sufficiently large odd numbers are representable in the form

$$n=p_1+p_2^2+p_3^3+p_4^4+p_5^5$$

with primes p_r, and that the number of such representations is of the true order of magnitude $N^\gamma/\log^5 N$ where

$$\gamma=1/2+1/3+1/4+1/5.$$

Some of the results of the first paper are used and in addition the major arcs are further divided. *L. Schoenfeld.*

Alder, Henry L. Generalizations of the Rogers-Ramanujan identities. Pacific J. Math. 4, 161–168 (1954).

The first Rogers-Ramanujan identity [Hardy and Wright, Introduction to the theory of numbers, Oxford, 1938, Theorem 362] states that the generating function for the number of partitions into parts of the form $5m\pm1$ is expressible as

$$\sum_{m=0}^{\infty} x^{m^2}/(1-x)(1-x^2)\cdots(1-x^m).$$

The author considers, for any positive integer k, the generating function for the number of partitions into parts that are not congruent to 0, k or $-k$ (mod $2k+1$), and shows that this function is expressible as

$$\sum_{m=0}^{\infty} G_{k,m}(x)/(1-x)(1-x^2)\cdots(1-x^m),$$

where $G_{k,m}(x)$ is a polynomial in x. The exact degree of this polynomial is determined in the cases $k=3$ and 4. A similar identity is obtained for the generating function for the number of partitions into parts that are not congruent to 0 or ±1 (mod $2k+1$), with $G_{k,m}(x)$ replaced by $x^mG_{k,m}(x)$. However, except in the original case of Rogers and Ramanujan, the series obtained are not interpretable as being themselves the generating functions for partitions into parts differing by at least a given amount. *H. Davenport.*

Dufresnoy, Jacques, et Pisot, Charles. Sur les petits éléments d'un ensemble remarquable d'entiers algébriques. C. R. Acad. Sci. Paris 238, 1551–1553 (1954).

Let S be the set of algebraic integers $\theta>1$ whose conjugates have all (except θ itself) moduli strictly inferior to 1. It is known that S is closed. By S' we denote the derived set. In their recent paper [Ann. Sci. Ecole Norm. Sup. (3) 70, 105–133 (1953); these Rev. 15, 605] the authors have determined the four smallest elements of S [the determination of the two smallest elements is due to C. L. Siegel, Duke Math. J. 11, 597–602 (1944); these Rev. 6, 39]. They have also proved the conjecture of C. L. Siegel, according to which the smallest element of S' is $\frac{1}{2}(1+\sqrt{5})$. In the present paper the authors determine all numbers of S inferior to $\frac{1}{2}(1+\sqrt{5})$, (and even all numbers of S inferior to a certain element $\omega>\frac{1}{2}(1+\sqrt{5})$). *R. Salem.*

Nöbauer, Wilfried. Über Gruppen von Restklassen nach Restpolynomidealen. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 162, 207–233 (1953).

Let Γ denote the ring of rational integers. A residual polynomial (mod n) is a polynomial $g(x)\in\Gamma[x]$ such that $g(a)\equiv0$ (mod n) for all integral a. Let A denote the ideal consisting of all residual polynomials (mod n). In the quotient ring $\Gamma[x]/A$ define an operation GF, where $F, G\in\Gamma[x]/A$, as follows. Let $f(x)\in F$, $g(x)\in G$; then the class GF is determined by the polynomial $f(g)$ mod A. The elements of $\Gamma[x]/A$ constitute a semigroup H_n with respect to this operation. The elements of H_n with inverses constitute a group G_n. For $F\in\Gamma[x]/A$ let $f(x)$ denote a polynomial of minimal degree $\in F$ and define $[F]=[f]=\deg f$. For given s let $\Delta_n(s)$ denote the number of $F\in G_n$ such that $[F]\leqq s$. The determination of this function is the main object of the paper. While this is not carried out completely, the following are some of the results obtained.

In the first place, if $n=ab$, where $(a,b)=1$, then G_n is the direct product of G_a and G_b; also $\Delta_{ab}(s)=\Delta_a(s)\Delta_b(s)$. For $s=1$ we have $\Delta_n(1)=n\phi(n)$, where $\phi(n)$ is the Euler function; moreover, $\Delta_n(1)\mid\Delta_n(s)$. For $n=p$, we have $\Delta_p(2)=\Delta_p(1)$ and $\Delta_p(3)$ is explicitly determined without much difficulty. For $n=p^e$ it is proved that the group $G_{p^{e+k}}$ is homomorphic to G_{p^e} for all $k\geqq0$. Let $\epsilon(i)$ denote the highest power of p dividing $i!$ and define r as the greatest integer such that $\epsilon(r)<e$. Then it is proved that for $e>2$, $s\leqq r$, $\Delta_{p^e}(s)=p^{r+1}\Delta_{p^{e-1}}(s)$. Thus the computation of $\Delta_{p^e}(s)$ is reduced to that of $\Delta_p(s)$ and $\Delta_{p^2}(s)$. The last part of the paper contains a number of results concerning G_{p^2} and $\Delta_{p^2}(s)$. The bound $\Delta_{p^2}(t)\leqq\Delta_p(t)p^{t+1}$ may be mentioned. The paper closes with some numerical data for $p=2, 3, 5, 7$, $e=1$.

L. Carlitz (Durham, N. C.).

Linnik, Yu. V. Some applications of Lobačevskiĭ's geometry to the theory of binary quadratic forms. Doklady Akad. Nauk SSSR (N.S.) 93, 973–974 (1953). (Russian)

The author announces some results on the distribution of the reduced positive definite binary quadratic forms $a\xi^2+2b\xi\eta+c\eta^2$ of large determinant $b^2-ac=-D<0$. On putting $2x=c+a$, $y=b$, $2z=c-a$, the reduced forms are represented by points (x,y,z) on the hyperboloid $x^2-y^2-z^2=D>0$ with y integral and x and z halves of integers of the same parity, in the region A defined by $2|y|\leqq x-z\leqq x+z$. Results are given concerning the number of such points in certain sub-regions of A. The method of proof is said to be similar to that indicated in a previous paper [Linnik and Malyšev, same Doklady (N.S.) 89, 209–211 (1953); these Rev. 15, 406], but to depend on the existence, in certain algebras of generalized quaternions with indefinite norm, of a ring of integers with a Euclidean algorithm. One of the results stated is as follows. Suppose $q\geqq3$ is a prime, and $(-D|q)=+1$. Then there exist forms (a,b,c) satisfying

$$\alpha_1D^{\frac{1}{2}}<a<\alpha_2D^{\frac{1}{2}},\quad \alpha_3D^{\frac{1}{2}}<b<\alpha_4D^{\frac{1}{2}},$$

where $\alpha_1,\cdots,\alpha_4$ are constants consistent with the conditions of reduction, provided $D>D_0(q,\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. Moreover, the number of such forms, divided by the total number $h(-D)$, is greater than a positive constant depending on $q,\alpha_1,\alpha_2,\alpha_3,\alpha_4$. *H. Davenport* (London).

Goldberg, Karl, Newman, Morris, Straus, E. G., and Swift, J. D. The representation of integers by binary quadratic rational forms. Arch. Math. 5, 12–18 (1954).

The authors consider the integers represented by

$$f(x, y)=(ax^2+bxy+cy^2)/(p+qxy),$$

where a, b, c, p, q are integers and without loss of generality $(p, q)=1$. They prove that (I) if there exist integers x_0, y_0 not both 0 such that $ax_0^2+bx_0y_0+cy_0^2=p+qx_0y_0=0$, $(ax_0, qy_0)|(c, qy_0)$ then $f(x, y)$ represents all the integers in some arithmetic progression, but (II) if $a|(b, q)$, $c|(b, q)$ then $f(x, y)$ represents only a finite number of integers with $xy\neq 0$. Here (I) results at once from the consideration of $f(x, y_0)$ and $f(x_0, y)$. The proof of (II) depends on the equivalence of $f(x, y)=m$ and $ax^2+Bxy+cy^2=mp$, $B=b-mq$, where $a|B$, $c|B$ by hypothesis. Hence the L.H.S. has the integral automorphs

$$x\rightarrow x,\quad y\rightarrow -\frac{B}{c}x-y$$

and

$$x\rightarrow -\frac{B}{a}y-x,\quad y\rightarrow y$$

whose product is of infinite order. Thus infinite descent applies (and the relevant Pellian equation has a solution which may easily be written down despite the authors' introduction). *J. W. S. Cassels* (Cambridge, England).

Rogers, C. A. The product of n non-homogeneous linear forms. Proc. London Math. Soc. (3) 4, 50–83 (1954).

Let x_i be n real linear forms in the n variables u_j and with determinant Δ, and let b_i be n real constants. Consider the inequalities

(1) $$|(x_1+b_1)\cdots(x_n+b_n)|<1,$$

(2) $$(x_1+b_1)^2+\cdots+(x_n+b_n)^2<\epsilon^2,$$

(3) $$x_1+b_1>0,\ \cdots,\ x_n+b_n>0,$$

(4) $$(x_1+b_1)\cdots(x_{n-1}+b_{n-1})<\epsilon^{n-1}.$$

Theorems, usually giving necessary and sufficient conditions, are proved for the existence of an infinity of solutions such that (1), (2) hold; or (1); or (1), (4); or (1), (2), (3). The proofs depend on slightly more general geometric theorems. The conditions used are natural ones. One of the basic conditions imposed is that the number $\lambda_1b_1+\cdots+\lambda_rb_r$ is an integer for every set of real numbers $\lambda_1, \cdots, \lambda_r$ such that the form $\lambda_1x_1+\cdots+\lambda_rx_r$ has integral coefficients for the u_j. A geometric equivalent is shown to be that there is a point of a grid (non-homogeneous lattice) Γ in the smallest linear manifold M which is generated by points of the lattice Λ of Γ and which contains the points satisfying $x_1=\cdots=x_r=0$. *L. Tornheim* (Ann Arbor, Mich.).

Černy, Karel. Sur les approximations diophantiennes. Čehoslovack. Mat. Ž. 2(77), 191–220 (1952). (Russian. French summary)

The author proves, with the aid of the theory of Lebesgue measure, a result of Jarník based on difficult considerations involving Hausdorff measure [Math. Z. 33, 505–543 (1931)]. Suppose that $s>1$, $\omega(x)$ is a positive continuous function for $x\geqq 1$, and $\omega^s(x)x^{s+1}$ is monotonic for $x\geqq 1$. Suppose also that $\int_1^\infty\omega^s(x)x^sdx$ and $\int_1^\infty\omega^{s-1}(x)x^{s-1}dx$ are respectively convergent and divergent. Finally, let $\tau(x)$ be a function defined for $x\geqq 1$ such that $\tau(x)/x\geqq 1$, and $\tau(x)/x\rightarrow\infty$ as $x\rightarrow\infty$. Then there exists a number Θ_s such that, for almost all points $(\Theta_1, \Theta_2, \cdots, \Theta_{s-1})$ of Euclidean $(s-1)$-dimensional space, $(\Theta_1, \Theta_2, \cdots, \Theta_s)$ is a proper system and admits the approximation $\omega(x)$ but not $\omega(\tau(x))$. Here $(\Theta_1, \Theta_2, \cdots, \Theta_s)$ is said to be proper if $k_0+k_1\Theta_1+k_2\Theta_2+\cdots+k_s\Theta_s=0$, for integral k_i, implies $k_0=k_1=\cdots=k_s=0$. In the proof Θ_s is constructed as a continued fraction whose convergents satisfy certain elaborate conditions.

From this result the following is deduced: Let $s\geqq 1$, and let $\omega(x)$, $\lambda(x)$ be two functions which are positive and continuous for $x\geqq 1$. Let the functions $\omega(x)x^2, \omega^2(x)x^3, \cdots, \omega^s(x)x^{s+1}, \omega(x)x^{1+\epsilon}$ (for a certain $\epsilon>0$), $\lambda(x)$ be monotonic and let $\lambda(x)\rightarrow 0$ for $x\rightarrow\infty$. Also suppose that $\int_1^\infty\omega^s(x)x^sdx$ is convergent. Then there exists a proper system

$$(\Theta_1, \Theta_2, \cdots, \Theta_s)$$

which admits the approximation $\omega(x)$, but not $\omega(x)\lambda(x)$. *R. A. Rankin* (Birmingham).

ANALYSIS

Motzkin, T. S. Two consequences of the transposition theorem on linear inequalities. Econometrica 19, 184–185 (1951).

The transposition theorem [Motzkin, Dissertation, Basel, 1933, Jerusalem, 1936] states that, of the two systems $Ax>0$, $Bx\geqq 0$, $Cx=0$ and $A'u+B'v+C'w=0$, $u\geqq 0$, $u\neq 0$, $v\geqq 0$, exactly one is solvable; here A, B, C are real matrices, prime denotes transpose, and vector inequalities hold in all components simultaneously. This note derives two direct consequences that are important in the theory of linear programming. I. If $Ax\geqq b$ is solvable and $c'x\geqq\delta$ for all of its solutions, then $A'y=c$, $b'y\geqq\delta$, $y\geqq 0$, is solvable. II. If δ is the minimum of $c'x$ subject to $Ax\geqq b$, $x\geqq 0$, then δ is the maximum of $b'y$ subject to $A'y\leqq c$, $y\geqq 0$. *H. W. Kuhn* (Bryn Mawr, Pa.).

Slater, Morton L. A note on Motzkin's transposition theorem. Econometrica 19, 185–187 (1951).

This note states the necessary generalization of the transposition theorem [see the preceding review] when a subset of the components of Bx is required to contain some positive components. A geometrical counterpart is proved in a special case and an application is indicated to the duality established for matrix problems with constraint equations by Gale, Kuhn, and Tucker [Activity analysis of production and allocation, Wiley, New York, 1951, pp. 317–329; these Rev. 13, 670]. *H. W. Kuhn* (Bryn Mawr, Pa.).

Rubinštein, G. Š. The general solution of a finite system of linear inequalities. Uspehi Matem. Nauk (N.S.) 9, no. 2(60), 171–177 (1954). (Russian)

Tripartite parametrical solution, identical with Motzkin, Dissertation, Basel, 1933, Jerusalem, 1936, p. 40. Also applications supplementing Černikov, Uspehi Matem. Nauk (N.S.) 8, no. 2(54), 7–73 (1953) [these Rev. 15, 293]. *T. S. Motzkin* (Los Angeles, Calif.).

Hayman, W. K., and Stewart, F. M. Real inequalities with applications to function theory. Proc. Cambridge Philos. Soc. 50, 250–260 (1954).

Part I of this paper is concerned with relations between the quantity $f_n(x)=\inf_{h>0}f(x+h)/h^n$ and the derivative $f^{(n)}(x)$. A function f is said to satisfy the hypothesis A_n if $f\geqq 0$, $f^{(n)}$ is continuous for all large x, and $f^{(n)}>0$ for some

arbitrarily large x. It is said to satisfy B_n, if together with its derivatives up to order $n-1$ it is non-negative, non-decreasing, and convex for $x\geqq 0$. The lower density of a set E on the positive axis is $\liminf_{r\to\infty} l(r)/r$, where $l(r)$ denotes the measure of E on $(0, r)$. If f satisfies A_n (satisfies B_n) then, for each $K>1$, $f_n(x)\leqq K(e/n)^n f^{(n)}(x)$ on a set of values x which is unbounded (has positive lower density). Here the inequality on $f^{(n)}(x)$ is to be taken as applying to both the left and right derivatives of $f^{(n-1)}(x)$. If f satisfies B_n and $f(x)\leqq\mu(x)$ for all $x\geqq 0$, then $f^{(n)}(x)\leqq n!\mu_n(x)$; if moreover μ satisfies A_n (satisfies B_n), then

$$f^{(n)}(x)\leqq Kn!(e/n)^n\mu^{(n)}(x)$$

on a set of values x which depends only on K $(K>1)$, n, and μ, but not on f, and which is unbounded (has positive lower density).

In Part II these results are applied to complex variable theory. Let f be entire, and let $\mu(r)$ satisfy A_n (satisfy B_n); if $|f(re^{i\theta})|\leqq\mu(r)$ for all $r\geqq 0$ and all θ, then, for each positive integer n and each $K>1$, the inequality

$$|f^{(n)}(re^{i\theta})|<Kn!(e/n)^n\mu^{(n)}(r)$$

holds on a set of r which is unbounded (has positive lower density). Other theorems concern the maximum modulus of entire functions, and the area of the map of $|z|<r$ upon the Riemann sphere by a meromorphic function. *G. Piranian.*

Ahiezer, N. I. On best weighted approximation on the whole axis by means of entire functions of finite degree. Doklady Akad. Nauk SSSR (N.S.) 94, 983–986 (1954). (Russian)

Let $\Phi(x)\geqq 1$ be a majorant of quasi-finite growth in S. Bernstein's terminology [same Doklady (N.S.) 65, 117–120 (1949); these Rev. 11, 23]. This means that for each positive p there is an entire function $\Phi_q(z)$ of exponential type $q=q(p)$ such that if $g(z)$ is entire and of exponential type p and $|g(x)|\leqq\Phi(x)$ for all real x then $|g(z)|\leqq|\Phi_q(z)|$ for $y\geqq 0$. Measure approximation to ϕ by ψ by means of the norm $\sup_{-\infty<x<\infty}|\phi(x)-\psi(x)|/\Phi(x)$. Let $f(x)$ be real and continuous, and let $G_p(x)$ be a real entire function of exponential type p. Let $\{f(x)-G_p(x)\}/\Phi(x)$ have a Chebyshev set [defined by the author, Mat. Sbornik N.S. 31(73), 415–438 (1952); these Rev. 14, 459], and let this set be the set of zeros of a real entire function $\Omega(z)$, of exponential type, for which $\lim y\Phi_q(iy)/\Omega(iy)=0$. Under these circumstances $G_p(x)$ is that entire function of exponential type p deviating least from $f(x)$ in the specified norm.

As an illustration the author takes $\Phi(x)$ to be the absolute value of an entire function $H(x)$ of exponential type and calculates an explicit formula for the best approximation to the function $(A+Bx)/(x^2+k^2)$ [obtained for $H(x)\equiv 1$ by Bernstein, same Doklady (N.S.) 51, 487–490 (1946); these Rev. 8, 20]. *R. P. Boas, Jr.* (Evanston, Ill.).

Pinsker, I. Š. On the construction of functions deviating least from zero. Doklady Akad. Nauk SSSR (N.S.) 95, 21–24 (1954). (Russian)

The classical theorems of Chebyshev on functions (of one real variable) of a given form which deviate least from zero have been extended to a much more general class than the original linear combinations formed from a prescribed sequence [see, e.g., Morozov, Izvestiya Akad. Nauk SSSR. Ser. Mat. 16, 75–100 (1952); these Rev. 13, 728]. Here the author lays down appropriate definitions and announces corresponding theorems for functions of several variables. *R. P. Boas, Jr.* (Evanston, Ill.).

Lyttkens, Sonja. The remainder in Tauberian theorems. Ark. Mat. 2, 575–588 (1954).

Let $F(x)\in BV[-\infty,\infty]$ and suppose that

$$f(x)\equiv\int_{-\infty}^{\infty}e^{ixt}dF(u)\neq 0$$

when x is real. Let E be the class of real bounded functions $\Phi(x)$ such that $\Phi(x)+e^{\epsilon x}$ is non-decreasing for every $\epsilon>0$ and $x>x_\epsilon$. Let $F(x)$ be an admissible Tauberian kernel; suppose the rate of approach of $\Psi(x)\equiv\int_{-\infty}^{\infty}\Phi(x-u)dF(u)$ to its limit A to be known; what can be said of the rate of approach of $\Phi(x)$ to A? Basing herself on earlier work of A. Beurling [Yhdeksäs Skandinaavinen Matemaatikkokongressi, Helsingfors, 1938, Mercator, Helsingfors, 1939, pp. 345–366] the author proves various results for the case in which $\Psi(x)-A=O(e^{-\gamma x})$, $\gamma>0$. Thus $\Phi(x)-A$ will be $O(e^{-\theta cx})$, $c=\min(a,\gamma)$, $\theta<2(2p+3)^{-1}$ if $[f(t)]^{-1}$ can be continued analytically into the strip $-a\leqq\Im(t)\leqq b$ and either $[f(t)]^{-1}$ is $O[(1+|t|)^p]$ or $\int_{-\infty}^{\infty}|f(x+i\beta)|^{-2}(1+|x|)^{-2p-1}dx=O(1)$, both uniformly in the strip. Similar relations hold if $[f(t)]^{-1}$ is holomorphic only in $-a\leqq\Im(t)<0$ provided

$$\frac{d}{dx}[f(x+i\beta)]^{-1}=O[(1+|x|)^{p-1}],\quad p>0,$$

or belongs to $L_2(-\infty,\infty)$ after multiplication by

$$(1+|x|)^{-2p+1}.$$

Further extensions hold if $[f(t)]^{-1}$ is holomorphic in a strip save for a finite number of poles. *E. Hille.*

Nudel'man, A. A. On the application of completely and absolutely monotone sequences to the problem of moments. Uspehi Matem. Nauk (N.S.) 8, no. 6(58), 119–124 (1953). (Russian)

The author gives an elementary discussion of the uniqueness of a solution of the moment problem on $(1,\infty)$,

$$\mu_n=\int_1^{\infty}t^n d\sigma(t),\quad n=0,1,2,\cdots;\ d\sigma(t)\geqq 0.$$

His method depends on a preliminary investigation of conditions under which a completely monotonic sequence $\{c_k\}_0^{\infty}$ can be extended to a completely monotonic sequence $\{c_k\}_{-\infty}^{\infty}$. *R. P. Boas, Jr.* (Evanston, Ill.).

Theory of Sets, Theory of Functions of Real Variables

Eyraud, Henri. Les récurrences des différentes classes. Cahiers Rhodaniens 5, 19–26 (1953).

✱Eyraud, Henri. Le théorème de la récurrence transfinie. Institut de Mathématiques, Université de Lyon, Lyon, 1953. iv+14 pp.

Move

These papers are intended to prove a conjecture made by the author [see, e.g., Comptes Rendus du Congrès des Sociétés Savantes de Paris et des Departements tenu à Grenoble en 1952, Section des Sciences, Gauthier-Villars, Paris, 1952, pp. 41–46; these Rev. 15, 409]. *F. Bagemihl* (Princeton, N. J.).

Riguet, Jacques. Systèmes de coordonnées relationnels. II. Applications à la théorie des groupes de Kaloujnine. C. R. Acad. Sci. Paris **238**, 435–437 (1954).

Riguet, Jacques. Systèmes de coordonnées relationnels. III. "τ", fermetures et systèmes symétriques. C. R. Acad. Sci. Paris **238**, 1763–1765 (1954).

These brief announcements summarize many formal properties of "relational coordinates" as defined by the author [cf. same C. R. **236**, 2369–2371 (1953); these Rev. **15**, 17], which can apparently only be expressed in the original vocabulary developed by the author. No proofs are given. *G. Birkhoff* (Cambridge, Mass.).

Farah, Edison. On the total order of the set of powers of the parts of a given set. Bol. Soc. Mat. São Paulo **5** (1950), 59–61 (1952). (Portuguese)

Using Zorn's theorem the author proves the comparability of cardinals [cf. also Hönig, Proc. Amer. Math. Soc. **5**, 312 (1954); these Rev. **15**, 690] and consequently that predecessors of any cardinal number form a chain (ramification condition of cardinals). *G. Kurepa* (Zagreb).

Shepherdson, J. C. On two problems of Kurepa. Pacific J. Math. **4**, 301–304 (1954).

Answering two of the reviewer's questions [Pacific J. Math. **2**, 323–326 (1952); these Rev. **14**, 255] the author proves independently from Gustin the existence of a ramified set F so that (1) no maximal antichain of F meets each maximal chain of F, (2) no maximal chain of F meets each maximal antichain of F. In particular, such a set F is the set of all finite sequences of integers ordered so that $(a_1, \cdots, a_m) \leqq (b_1, \cdots, b_n)$ means $m \leqq n$, $a_i = b_i$ $(i < m-1)$, $a_m \leqq b_m$. [As to the second question cf. also the sets σ_0, p. 95, and σ, p. 102 in the reviewer's thesis, Paris, 1935].
G. Kurepa (Zagreb).

Kurepa, G. On some coupled operators in order sets. Bull. Soc. Math. Phys. Serbie **5**, no. 3–4, 15–21 (1953). (Serbo-Croatian summary)

Let S be any partially ordered set. Denote by OS and by $\bar{O}S$ the system of all maximal chains and maximal sets of incomparable elements of S respectively. Let f be a mapping of S into a family of subsets of some set Q. For F in $\{OS, \bar{O}S\}$ let $(F, \bigcap, f) = \bigcup_{X \in F} \bigcap_{x \in X} f(x)$. The author seeks relations between $(\bar{O}S, \bigcap, f')$ and $(OS, \bigcap, f)'$, where $'$ denotes complementation. Given S, a necessary and sufficient condition that $(\bar{O}S, \bigcap, f')$ is a subset of $(OS, \bigcap, f)'$ for each f is that $A \cap M$ be non-empty for each A in $\bar{O}S$ and each M in OS. The reviewer is unable to understand Theorem 3.1, the two statements comprising it appearing to contradict each other. The reading of the paper is hampered by usage of undefined symbols and terms, and by errata. For example, $(3.1)_2$ should be $(\bar{O}S, \bigcap, f')$; and part of line 2, page 19 should be $-n, n+1, n+2, \cdots$. *S. Ginsburg.*

Berge, Claude. Sur les ensembles purs et les ultrafiltres. C. R. Acad. Sci. Paris **238**, 2136–2137 (1954).

Suppose that f is a multivalent mapping of a space E into a space E'. Let us define the strong inverse $f^{(-1)}$ by $f^{(-1)}(S) = \{x : f(x) \subset S\}$. The author calls a set pure if its inverse in the usual sense is equal to its strong inverse. The pure sets form a complete field and a necessary and sufficient condition that every filter on E' generated by an ultrafilter on E decide for or against A is that A be a pure set.
H. Rubin (Stanford, Calif.).

Glivenko, E. V. On sets of values of additive vector-functions. Mat. Sbornik N.S. **34**(76), 407–416 (1954). (Russian)

Čuĭkina [Doklady Akad. Nauk SSSR (N.S.) **76**, 801–804 (1951); these Rev. **12**, 486] proved that a subset of n-dimensional Euclidean space is the range of an additive vector-valued set-function if and only if that set is the limit (in the sense of the Hausdorff metric) of a sequence of finitely-generated convex sets whose sets of generators form an increasing sequence. The author removes the latter proviso by proving that an arbitrary limit of finitely-generated convex sets is also the limit of a sequence satisfying that monotory condition. *P. R. Halmos* (Chicago, Ill.).

***Aumann, Georg. Reelle Funktionen.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd LXVIII. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954. viii+416 pp. DM 59.60.

The main body of the book is divided into nine chapters; in what follows the contents and the flavor of each of the chapters will be briefly described. Chapter 1 treats general set theory from the mathematician's usual (non-axiomatic) point of view. The basic set-theoretic operations are described and the notation is established; countable sets and sets of the power of the continuum are discussed. Chapter 2 is longer and meatier; the subject is the general concept of (partial) order. It includes MacNeille's theory of completion by cuts, totally ordered sets, the characterization of the set of real numbers as a complete ordered field, well ordered sets, Zorn's lemma and the well-ordering theorem, ordinal and cardinal numbers, the Borel and Suslin operations, and the convergence theory of functions on directed sets. Chapter 3 specializes the discussion to lattices and Boolean algebras; it includes a proof of the representation theorems associated with the names of Stone and Loomis. Chapter 4 treats some topological concepts. In addition to the usual basic concepts (closure, density, compactness), the chapter discusses the Boolean algebra of the so-called regularly open sets, Peano curves, the characterization of topology by convergence, and sets possessing the property of Baire.

The emphasis in the next three chapters shifts to the theory of real-valued functions. In Chapter 5, for instance, normal spaces are defined and the problem of metrization is discussed. Also, in the same chapter, semicontinuous functions and Baire functions are treated, the Stone-Weierstrass theorem is proved, the fixed-point theorem concerning contractions in complete metric spaces is proved and applied to the implicit function theorem, and the Brouwer fixed-point theorem is proved and applied to an n-dimensional generalization of Bolzano's intermediate-value theorem. Chapter 6 treats metrics in Cartesian products of metric spaces and the connections between separate continuity and joint continuity. Chapter 7 treats the theory of differentiation and integration of a real-valued function of a single real variable. Bounded variation and absolute continuity are discussed; the almost everywhere differentiability of a monotone function is proved by Riesz's method; Perron, Riemann, Lebesgue, and Stieltjes integration are developed; and the fundamental theorem of the integral calculus is proved.

The last two chapters are devoted to abstract measures and integrals. Measure theory is systematically treated in Chapter 8 from the Boolean-algebraic point of view. The treatment includes a generalized Burkill integration theory.

Abstract integration appears in Chapter 9 (whose title is "Positive linear functionals"); the treatment (including the Radon-Nikodym theorem and the Fubini theorem) is completely along the lines developed by Stone.

Although in principle the book is self-contained, a student should be advised to study it only if he has had a more than usually rigorous training in calculus. Another prerequisite is the willingness to read line by line, with no skipping; the author's style is terse, sometimes to the point of being telegraphic. (The Cantor-Bernstein theorem on cardinal numbers is proved in twelve lines.) Another factor that makes for slow reading is the author's fondness for alphabetic notation; the adjective "ordered" might have as a prefix t, s, k, w, or i (referring to the initials of the German words for partial, weak, chain, well, and inductive, respectively). There are other notational aspects that might slow down a mathematician already set in his ways more than a beginning student. Thus implication is symbolized by a roughly equilateral triangle (with an apex pointing to the consequent), and logical equivalence is symbolized by two such triangles placed to look like a figure eight with corners. The symbol for symmetric difference is a plus sign with a dot under it (called "contraplus"); this is to be contrasted with the plus sign with a dot over it (called "coplus") symbolizing union. The closure and the derived set of a set A are denoted by A^α and A^β respectively. There are some corresponding instances of terminological unorthodoxy. Thus the definition of a general topological space includes the condition that a one-point set be closed; a space satisfying the second axiom of countability is called a rational space; and a Cauchy sequence is called a concentrated sequence.

The book packs an amazing amount of valuable material into relatively little space. It might seem unfair, therefore, to accuse the author of sins of omission. Nevertheless it seems to the reviewer that a slight expansion of the treatment of ordinal and cardinal numbers would have made the author's definitions much more satisfying, and, in the topological chapter, the theories of connectedness and Cartesian products (especially for compact spaces) would have been most useful additions. These are, however, minor criticisms. The volume is physically attractive and beautifully printed; the quality, quantity, organization, and exposition of its contents, together with the fact that much of the material in it has not been available hitherto in book form, serve to make it a recommended part of the library of every modern analyst. *P. R. Halmos* (Chicago, Ill.).

Blackwell, David. A representation problem. Proc. Amer. Math. Soc. 5, 283–287 (1954).

Let f be a non-negative integrable function on a non-atomic measure space and a a fixed number less than the measure of the space. Then a necessary and sufficient condition that there exist non-negative numbers c_n whose sum converges and functions φ_n each of which is a characteristic function of a set of measure a such that $f(x)=\sum_1^\infty c_n\varphi_n(x)$ is that $f(x)\leqq a^{-1}\int f(x)dx$ for all x. The sufficiency is proved by decreasing f by functions of the form $c\varphi$. *H. Rubin.*

Green, John W. Note on the smoothness of integral means. Arch. Math. 5, 53–55 (1954).

It is proved that if g is a measurable function such that $g(x+h)-g(x)$ is a continuous function of x for each fixed h, then g is continuous. It is deduced that if f is summable and M_h $(M_h(x)=h^{-1}\int_x^{x+h}f(t)dt)$ has a continuous derivative, then f is equal almost everywhere to a continuous function. A similar result involving higher derivatives of M_h is obtained using a related theorem of L. D. Thompson [Proc. Amer. Math. Soc. 4, 402–407 (1953); these Rev. 14, 957]. *F. F. Bonsall* (Newcastle-upon-Tyne).

Medvedev, Yu. T. Generalization of a theorem of F. Riesz. Uspehi Matem. Nauk (N.S.) 8, no. 6(58), 115–118 (1953). (Russian)

F. Riesz [Riesz and Nagy, Leçons d'analyse fonctionelle, Akad. Kiadó, Budapest, 1952, p. 75; these Rev. 14, 286; Riesz, Math. Ann. 69, 449–497 (1910)] gave a condition necessary and sufficient that a function on a closed interval be the indefinite integral of a function in L_p, $p>1$. This note extends this theorem to the spaces L_M of Birnbaum and Orlicz [Studia Math. 3, 1–67 (1931)], when $M(u)/u\to\infty$ as $u\to\infty$. *M. M. Day* (Urbana, Ill.).

Becisey, E. M. Concerning total differentiability of functions of class P. Pacific J. Math. 4, 169–205 (1954).

The author considers the total differentiability of functions of the class P of functions of two real variables which are of bounded variation in the sense of Pierpont. Burkill and Haslam-Jones proved [J. London Math. Soc. 7, 297–305 (1932)] that, if f belongs to the class A of functions of bounded variation in the sense of Arzelà, then f is totally differentiable almost everywhere. Adams and Clarkson [Trans. Amer. Math. Soc. 35, 824–854 (1933); 36, 711–730 (1934); 46, 468 (1939); these Rev. 1, 48] proved that, if $\mathfrak{T}$ denotes the class of functions of bounded variation in an extended Tonelli sense and M is the class of measurable functions, then $\mathfrak{T}\cap M\supset P\supset A$, and each function in $\mathfrak{T}\cap M$ is approximately differentiable almost everywhere: Saks had previously shown [Ann. of Math. (2) 34, 114–124 (1933)] that there exist functions in $\mathfrak{T}\cap M$ which are nowhere totally differentiable, and that, with a suitable norm in the function space, the set of such functions is residual in a certain sub-class E of $\mathfrak{T}\cap M$. In the present paper an example is given of a continuous function in P which is nowhere totally differentiable. It is further shown that, with the usual norm in the space of continuous functions, the set of such non-differentiable functions is residual in the set of continuous functions of class P, and that (with a different norm) it is also residual in the set $P\cap E$. Finally it is shown that $P\cap E$ is of the first category in E. *U. S. Haslam-Jones* (Oxford).

Young, L. C. On the compactness and closure of surfaces of finite area, continuous or otherwise, and on generalized surfaces. Rend. Circ. Mat. Palermo (2) 2, 106–118 (1953).

Let R be the unit square $0\leqq u\leqq 1$, $0\leqq v\leqq 1$ in the $w=u+iv$ plane, and let $f(w)$ be a function in R whose values are points in Euclidean n-space. For each subset W of R, denote by $I(W,f)$, $a(W,f)$ the integrals over W of $\frac{1}{2}(f_u^2+f_v^2)$ and $[f_u^2f_v^2-(f_uf_v)^2]^{1/2}$ respectively (assuming that these integrals exist). Let N be a finite positive constant, to be kept fixed throughout. Then $f(w)$ is termed a generalized Dirichlet representation (N) if the following holds. (a) f is absolutely continuous with respect to v on almost every line u=constant, and absolutely continuous with respect to u on almost every line v=constant. (b) f is continuous on the perimeter of R. (c) $I(R,f)<N$. If f is continuous in R and satisfies (a), (b), (c), then f is termed a special Dirichlet representation (N). Finally, if f satisfies (a), (b), and $I(R,f)\leqq N$, then f is said to be a special or generalized

Dirichlet representation $(N+)$, according as f is or is not continuous in R. One of the main results in the paper is the following selection theorem. Let f_n be a sequence of generalized Dirichlet representations (N) which converge uniformly on the perimeter P of R. Then there exists a subsequence g_n of f_n and a function g which is a generalized Dirichlet representation $(N+)$, such that g_n converges to g in the following sense: There exists a dense set U in $0\leqq u\leqq 1$ and a dense set V in $0\leqq v\leqq 1$ such that (i) if $u_0 \in U$ and $v_0 \in V$, then $g_n(u_0, v_0)$ converges to $g(u_0, v_0)$, and (ii) if $u_0 \in U$, then on the line $u=u_0$ the functions g_n, g are absolutely continuous with respect to v and the integrals of g^2_{nv}, taken over $0\leqq v\leqq 1$, are uniformly bounded, with an analogous requirement for the lines $v=v_0 \in V$. Another main result of the paper has the character of an approximation theorem. Given a function f which is a generalized Dirichlet representation (N), let $\epsilon>0$ be assigned. Then there exists a function g which is a special Dirichlet representation (N) and which is within ϵ of f in the following sense: there exists an open set W in R such that the measure of W is less than ϵ, $a(W, f)<\epsilon$, $a(W, g)<\epsilon$, and finally $f=g$ on $R-W$. The paper concludes with the remark that various applications of the results of the present paper to generalized surfaces will be published later.

T. Radó (Columbus, Ohio).

Theory of Functions of Complex Variables

on order

*Valiron, Georges. **Fonctions analytiques.** Presses Universitaires de France, Paris, 1954. 236 pp. 1500 francs.

The monograph under review is concerned with selected topics in the theory of analytic functions. It presupposes merely the elements of the theory (which are summarized in a first preliminary chapter); autonomy of exposition is sought as well as the avoidance of duplication with other well-known monographs on the theory of functions. Emphasis is placed on expounding the more elementary parts of the theories studied. An abundance of references points the way to further detailed study of the topics treated. The geometric approach is stressed. In particular, insistence is laid upon the primordial importance of the notions of analytic continuation and the Riemannian image of an analytic function. Chapter 1 summarizes the principal notions of the theory of analytic functions up to analytic continuation. The Riemann mapping theorem is cited without proof. A number of results of Schwarz are given, including the determination of the mapping function for the interior of an ellipse. Chapter 2 treats topics in the theory of univalent functions (theorems of Bieberbach and Faber, the Koebe distortion theorem, Littlewood's theorem on the coefficients of a univalent function). Chapter 3 treats two theorems of Fatou. The first is the convergence theorem of Fatou and M. Riesz concerning power series with radius of convergence one having coefficients which form a null sequence. Application is made to a theorem of Poincaré concerning analytic functions with natural boundaries. The second theorem of Fatou considered is the celebrated radial-limit theorem. The chapter terminates with the well-known theorem of F. and M. Riesz. Chapter 4 is concerned with the boundary behavior of conformal maps. The notion of angular derivative is developed. Chapters 5 and 6 are concerned with the theory of iteration of analytic functions. A detailed study is made in Chapter 5 of the Fatou functions ((1, n) conformal maps of the interior of unit circle onto itself). Analytic functions mapping the right-half plane onto itself are studied in Chapter 6. Chapter 7 treats Poincaré functions admitting a multiplication theorem. Chapter 8 is concerned with certain classes of analytic functions which are non-continuable. The construction due to Valiron of a hyperbolic covering surface of the finite plane with a single logarithmic ramification point over infinity and no other transcendental critical points is given. In Chapter 9 the methods of Wiman and Valiron are developed and applied. Chapter 10 studies the growth of the solutions of algebraic differential equations. The above summary indicates the wealth of material considered. The exposition is lucid. This book should be of considerable value for continuing courses in the theory of analytic functions. *M. Heins.*

Brunk, H. D. On the growth of functions having poles or zeros on the positive real axis. Pacific J. Math. **4**, 1–19 (1954).

Let Δ denote a region containing a right half of the real axis, and let $f(z)$ be a function holomorphic in Δ except possibly for poles on the positive real axis. The author applies Mandelbrojt's results on adherent Dirichlet series to study the growth of f. This growth is related to the growth of leading coefficients in the Laurent developments about the zeros or poles of f in Δ (zeros or poles in λ_n, where $0<\lambda_n \uparrow \infty$). For that purpose he studies the integral $(2\pi i)^{-1}\int_\Gamma f(s)e^{-ss}ds$, where Γ is the boundary of Δ.

S. Mandelbrojt (Houston, Tex.).

Khanna, Girja. On a theorem of Phragmén-Lindelöf. Proc. Nat. Acad. Sci. India. Sect. A. **21**, 225–227 (1952).

The author shows, in effect, that

$$h(x)=\limsup_{|y|\to\infty} |y|^{-1}\log|f(x+iy)|$$

is convex for functions which are analytic in a vertical strip and satisfy an appropriate restriction on their rate of growth.

R. P. Boas, Jr. (Evanston, Ill.).

Khanna, Girja. Two theorems concerning analytic function of an analytic function. Proc. Nat. Acad. Sci. India. Sect. A. **21**, 228–230 (1952).

The author proves two uniqueness theorems, but the significance of the "function of a function" feature was not clear to the reviewer. The theorems appear to be corollaries, one of Carlson's theorem [about $f(n)=0$]; the other, of S.P. Jain's theorem [same Proc. **4**, 37–38 (1934)] that a function which is analytic and of exponential type in a strip $x\geqq -\delta$, $|y|<c$, vanishes identically if its Laplace transform $F(\sigma+it)$ has infinitely many zeros in a half plane $\sigma>\eta>0$.

R. P. Boas, Jr. (Evanston, Ill.).

Ahmad, Mansoor. On exceptional values of entire functions of infinite order. J. Indian Math. Soc. (N.S.) **18**, 19–21 (1954).

The author defines the kth order (k a positive integer) of an entire function $f(z)$ by

$$\rho_k=\limsup_{r\to\infty}\frac{l_{k+1}M(r)}{\log r}$$

where l_p denotes the p times iterated logarithm. Let $f(z)$ be entire, of infinite kth order but of finite $(k+1)$th order. The author shows that there exists at most one entire function $f_1(z)$ of finite kth order such that the roots of the equation $f(z)=f_1(z)$ are the zeros of an entire function of finite kth order. Reviewer's remarks: The hypothesis that

$f(z)$ be of finite $(k+1)$th order is probably superfluous. The case $k=1$ is due to Borel [Leçons sur les fonctions entières, 2e éd., Gauthier-Villars, Paris, 1921, p. 102].
J. Korevaar (Madison, Wis.).

Leont'ev, A. F. On overconvergence of a series. Doklady Akad. Nauk SSSR (N.S.) **94**, 381–384 (1954). (Russian)

Ostrowski proved [Abh. Math. Sem. Hamburg. Univ. **1**, 327–350 (1922), see p. 335] that if a power series $\sum a_n z^{\lambda_n}$ has very large gaps $(\lambda_{n_i+1}/\lambda_{n_i}\to\infty)$, it overconverges uniformly throughout the domain of regularity of the function which it represents. Under certain hypotheses which cannot be stated here, the author proves an analogous proposition concerning series of the form $\sum a_{n_i} f(\lambda_i z)$, where $f(z)=\sum a_n z^n$ is an entire function of finite order. *G. Piranian.*

Reich, Edgar. An inequality for subordinate analytic functions. Pacific J. Math. **4**, 259–274 (1954).

Let $f(z)=\sum_0^\infty c_n z^n$, $F(z)=\sum_0^\infty C_n z^n$ be regular in $|z|<1$ and $f(z)$ subordinate to $F(z)$. Let

$$a(r)=\pi\sum_1^\infty n|c_n|^2 r^{2n}=\int\int_{|z|<r}|f'(z)|^2|dz|^2$$

and let $A(r)=\pi\sum_1^\infty n|C_n|^2r^{2n}$. Then the author proves that

$$a(r)\leqq A(r)\sup_n nr^{2n-2},\quad 0<r<1,$$

and that this inequality is sharp for $0<r<1$. All cases of possible equality are given. The special case $a(r)\leqq A(r)$, $r\leqq 2^{-1/2}$ is due to Goluzin [Mat. Sbornik N.S. **29**(71), 209–224 (1951); these Rev. **13**, 223]. *W. K. Hayman.*

Scholz, D. R. Some minimum problems in the theory of functions. Pacific J. Math. **4**, 275–299 (1954).

The paper is concerned with extremal problems of the type

$$\int\int_D |f'(z)|^2 dxdy=\lambda=\text{minimum},$$

$$(1)\qquad \int\int_D \sigma(z)|f(z)|^2dxdy=1,\quad \sigma(z)>0,$$

where $f(z)$ is analytic and single-valued in a given domain D. In order to exclude constant solutions, it is further supposed that either

$$f(a)=0,\quad a\in D,\quad\text{or}\quad \int\int_D\sigma(z)f(z)dxdy=0.$$

The minimum of (1) is denoted by λ_1, and λ_n $(n=2, 3, \cdots)$ denote the successive minima of (1) under the additional conditions

$$\int\int_D\sigma(z)f(z)\overline{f_k(z)}dxdy=0$$
$$(k=1, 2, \cdots, n-1;\ f_k=k\text{th eigenfunction}).$$

It is first proved that the eigenfunctions $f_n(z)$ constitute a complete orthonormal set of functions whose first derivatives are also orthogonal over D and complete in the space of functions with single-valued integrals and finite norm. If the eigenvalues are not multiple, this system is unique up to a factor of unit absolute value.

Putting $\sigma\equiv 1$, the author further shows that, under condition $f(a)=0$, the eigenvalues and eigenfunctions are identical with those of the integral equation

$$(2)\qquad f(w)=\int\int_D \overline{K(z, w, a)}f(z)dxdy,$$

where the kernel is constructed from the well-known Bergman kernel $K(z, w)$ by means of the integral

$$K(z, w, a)=\int_a^w\int_a^z K(z, w)dzd\bar{w}.$$

With the help of Schiffer's theory, the variation of the eigenvalues is studied when the domain D is subjected to a first-order normal variation. Having treated the inhomogeneous equation corresponding to (2), the author derives a boundary relation and uses it, in some cases, for the effective computation of the functions $f_n(z)$. *O. Lehto.*

Keogh, F. R. A property of bounded schlicht functions. J. London Math. Soc. **29**, 379–382 (1954).

Let $f(z)$ be regular, univalent, and bounded in $|z|<1$, and let $l(\rho)$ be the length of the image curve of the segment $0\leqq z\leqq\rho<1$. Then $l(\rho)=o((-\log(1-\rho))^{1/2})$ and the exponent 1/2 is best possible. *A. W. Goodman.*

Mathur, Yogendra Behari Lal. On exceptional values of meromorphic functions. I, II, III, IV. Proc. Nat. Acad. Sci. India. Sect. A. **21**, 213–216, 217–219, 220–223, 224 (1952).

Dans cette série de notes l'auteur s'est afforcé d'apporter divers compléments à des travaux précédents sur la théorie de la distribution des valeurs exceptionnelles des fonctions méromorphes se rattachant à divers travaux de Collingwood et aux travaux antérieurs de Gross-Iverson-Dinghas-Laurent Schwartz-Nevanlinna-Parreau-Selberg-Teichmüller-Titchmarsh-Tumura-Yosida. Nous nous bornerons ici à rappeler le résultat principal de Collingwood [C. R. Acad. Sci. Paris **179**, 1125–1127 (1924); Trans. Amer. Math. Soc. **66**, 308–346 (1944); ces Rev. **11**, 94]. Si $f(z)$ est méromorphe dans le cercle de rayon R fini ou infini et si pour une valeur de a, il y a un $\sigma(r)$ positif, soit constant, soit tendant de façon monotone vers zéro quand r tend vers R et $ap(r)$ vérifie $0\leqq p(r)<\infty$. Pour tout $r<R$ ayant un point limite dans le domaine $E(a, \sigma(r), p(r))$ nous avons alors

$$m(r, a)<(\pi+\log r)p(r)+\log\frac{1}{\sigma(r)}+O(1)$$

dans le cas parabolique $R=\infty$ et

$$m(r, a)<[\pi+O(1)]p(r)+\log^+\frac{1}{\sigma(r)}+O(1)$$

dans le cas hyperbolique $R<\infty$.

L'auteur a recherché une généralisation de ce théorème en ajoutant aux hypothèses diverses conditions, puis s'est efforcé de voir si ces conditions ne pouvaient pas être supprimées ou affaiblies. D'autre part il a essayé de généraliser le lemme d'Iversen pour des fonctions situées dans le cercle unité. Sa démonstration fait appel aux notions de valence et de niveau. *G. Valiron* (Paris).

Wittich, H. Einige Eigenschaften der Lösungen von $w'=a(z)+b(z)w+c(z)w^2$. Arch. Math. **5**, 226–232 (1954).

The properties discussed relate to order and distribution of values as considered in the Nevanlinna theory of integral and meromorphic functions. $a(z)$, $b(z)$ and $c(z)$ are polynomials or rational fractions and the solutions considered are meromorphic in the whole plane. The proofs are made very simple by the employment of a considerable variety of methods. This enables some classical results to be included and it is shown further, that apart from rational solutions

(examples given), the order must be finite but at least $\frac{1}{2}$ and the distribution of values extremely regular.
A. J. Macintyre (Aberdeen).

Collingwood, Edward F. Sur les ensembles d'accumulation radiaux et angulaires des fonctions analytiques. C. R. Acad. Sci. Paris **238**, 1769–1771 (1954).

Dans cette note E. F. Collingwood emploie la terminologie et les notations de son mémoire des Acta Math. **87**, 83–146 (1952) [ces Rev. **14**, 260]. Nous ne les reproduirons pas ici. L'auteur dit qu'un arc α de la circonférence $|z|=1$ est un arc de Fatou pour une fonction méromorphe $f(z)$. Si c'est un arc accessible de la frontière d'un domaine simplement connexe contenu dans $|z|<1$ et pour lequel $f(z)$ est bornée ou si $1/[f(z)-a]$ est bornée pour une certaine valeur $a\neq\infty$. Il utilise deux lemmes dont la démonstration est donnée en détail. De ces lemmes il déduit plusieurs théorèmes. 1) Si $f(z)$ est méromorphe dans $|z|<1$ et $I(f)$ dense sur un arc α de $|z|=1$, $I(f)\cap\alpha$ est résiduel sur a. 2) Si $f(z)$ est méromorphe dans $|z|<1$, la différence entre les ensembles $S(f)$ et $I(f)$ est un ensemble de première catégorie sur la circonférence $|z|=1$. 3) Si $f(z)$ est méromorphe dans $|z|<1$ et si un des ensembles $S(f)$ ou $I(f)$ est dense sur un arc α de la circonference $|z|=1$, leur intersection est résiduelle sur α.
G. Valiron (Paris).

Meiman, N. N. Additions and corrections to the paper, "Differential inequalities and some questions of the distribution of zeros of entire and single-valued analytic functions". Uspehi Matem. Nauk (N.S.) **8**, no. 6(58), 177–180 (1953). (Russian)

The paper appeared in the same journal (N.S.) **7**, no. 3(49), 3–62 (1952); these Rev. **14**, 259.

Denjoy, Arnaud. Calcul approché des zéros de certaines fonctions entières dont on connait un développement asymptotique. C. R. Acad. Sci. Paris **238**, 1849–1853 (1954).

Denjoy, Arnaud. Application à la fonction $\zeta(s)$ du calcul approché des zéros. C. R. Acad. Sci. Paris **238**, 1945–1948 (1954).

In a preceding note [same vol., 1077–1080 (1954); these Rev. **15**, 694] the author discussed the asymptotic expansion of an entire function with real negative zeros. Here he discusses the inverse problem: given that the zeros are real and negative, and given an asymptotic expansion, determine the location of the zeros. The second note considers the possibility of applying the author's method to the zeta function (the author is not too hopeful), and suggests a number of auxiliary problems.
R. P. Boas, Jr.

Popov, B. S. Quelques propriétés des fonctions d'une variable complexe. Bull. Soc. Math. Phys. Macédoine **4** (1953), 20–24 (1954). (Macedonian summary)

The author describes what happens to the angle between two curves in the z-plane under a mapping

$$w=w(z)=u(x,y)+iv(x,y)$$

whenever $w(z)$ reduces to one of the special cases:

$$(1)\qquad w(z)=u(x)+iv(y);$$

$$(2)\qquad w(z)=A(x)+i[yA'(x)+B(x)].$$

As a geometric application, a relation is found between these special cases and the vectorial measure of deflection of Bilimovitch [C. R. Acad. Sci. Paris **237**, 694–695 (1953); these Rev. **15**, 521].
A. J. Lohwater.

Tsuji, Masatsugu. On covering surfaces of a closed Riemann surface of genus $p\geq 2$. Tôhoku Math. J. (2) **5**, 185–188 (1953).

In a previous paper [Nagoya Math. J. **6**, 137–150 (1953); these Rev. **15**, 518] the author studied certain smooth unbounded covering surfaces of a closed Riemann surface F of genus $p\geq 2$. Concerning the covering surfaces there designated $F_q^{(\infty)}$ $(q=1,\cdots,p)$, the author previously showed that $F_1^{(\infty)}$ has a null boundary but $F_q^{(\infty)}$ $(q\geq 2)$ has a positive boundary. In the present paper it is shown that the Green's function $G(z,z_0)$ of $F_q^{(\infty)}$ $(q\geq 2)$ tends to zero as z tends to the ideal boundary of $F_q^{(\infty)}$.
M. Heins.

***Calabi, E. Metric Riemann surfaces.** Contributions to the theory of Riemann surfaces, pp. 77–85. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

Il s'agit de caractériser toutes les métriques qui peuvent être induites sur une surface de Riemann par immersion analytique et régulière dans l'espace unitaire C^N à N dimensions $(N\leq+\infty)$. Cette note est une spécialisation, au cas des variétés à deux dimensions réelles, d'un article de l'auteur [Ann. of Math. (2) **58**, 1–23 (1953); ces Rev. **15**, 160]. En outre, on obtient les fonctions holomorphes définissant l'immersion, dans C^N, d'une surface de Riemann $\mathfrak{M}$ munie d'une métrique, lorsque les conditions nécessaires sont satisfaites et on prouve que toutes les courbures de $\mathfrak{M}$ (définies comme pour une courbe réelle dans l'espace euclidien) peuvent être calculées à partir de la métrique.
P. Dolbeault (Paris).

Rankin, R. A. On horocyclic groups. Proc. London Math. Soc. (3) **4**, 219–234 (1954).

Let $\bar\Gamma$ be a group of 2×2 matrices $T=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ with complex coefficients satisfying $ad-bc=1$, let Γ denote the corresponding inhomogeneous group; in particular, let $\Gamma(1)$ stand for the full modular group and $\Gamma(N)$ for the principal congruence subgroup (mod N). If $\bar\Gamma$ is properly discontinuous and leaves some circle $\mathfrak{S}$ invariant, $\bar\Gamma$ (or Γ) is called Fuchsian. If $\mathfrak{S}$ is identical with the set of limit points of the centers of the isometric circles, the Fuchsian group Γ is said to be of the first kind, otherwise of the second kind; the author uses the term "horocyclic" for Fuchsian groups of the first kind. In the particular case when Γ contains parabolic substitutions there is no loss in generality in taking ∞ as a parabolic point and $\mathfrak{S}$ as the real axis. Such a group is called zonal horocyclic; it has (some of its) fundamental regions $D(\Gamma)$ bounded by two vertical semi-infinite bounding lines and a (finite, or infinite) number of bounding arcs. The non-Euclidean area of $D(\Gamma)$ is defined by $J(\Gamma)=\iint_{D(\Gamma)}y^{-2}dxdy$; $J(\Gamma)$ is an invariant of the group and is finite if and only if $D(\Gamma)$ is bounded by a finite number N of bounding arcs [see Tsuji, Jap. J. Math. **21**, 1–27 (1952); these Rev. **14**, 968]. If N_1 is the number of cusps and N_2 the number of vertices of $D(\Gamma)$, then $N=N_1+N_2$. Using Ford's method of isometric circles [Automorphic functions, McGraw-Hill, New York, 1929] and results of Petersson [Math. Ann. **115**, 23–67 (1937)] and Poincaré [Acta Math. **3**, 49–92 (1893)], the author proves the identity of the horocyclic groups with the "Grenzkreisgruppen", as defined by Petersson [loc. cit.]. Furthermore, if Γ^* is Fuchsian and Γ is a subgroup of finite index μ of Γ^*, then Γ is Fuchsian and of the same kind as Γ^*. If Γ^* is a zonal horocyclic group, then $D(\Gamma)$ has a finite, or infinite

number of sides, according as $D(\Gamma^*)$ has. The inequalities $N\pi \geqq J(\Gamma) \geqq N_1\pi + N_2\pi/3 \geqq (N_1+N_2)\pi/3 = N\pi/3$ are established and used to prove that

$$\mu\pi^{-1}J(\Gamma^*) \leqq N \leqq 3\mu\pi^{-1}J(\Gamma^*) \leqq 3\mu N^*,$$

where N^* and N are the number of arcs of isometric circles bounding $D(\Gamma^*)$ and $D(\Gamma)$, respectively. In particular, if $\Gamma^* = \Gamma(1)$, then $J(\Gamma^*) = \pi/3$ and $\mu/3 \leqq N \leqq \mu$; if $\Gamma^* \neq \Gamma(1)$, $J(\Gamma^*) > \pi/3$. An upper bound is established for the radii of the isometric circles bounding $D(\Gamma)$, where Γ is a subgroup of $\Gamma(1)$; for $\Gamma = \Gamma(N)$ the bound takes the simple form $|c| \leqq N^3(N+1)^{\frac{1}{2}}(N+2)^3/8$. *E. Grosswald.*

Hervé, Michel. Fonctions fuchsiennes relatives à un groupe d'automorphismes du bicercle-unité. J. Analyse Math. **3**, 59–80 (1954).

The author gives a detailed account of the theory of Fuchsian functions of two variables [cf. Hervé, Ann. Sci. Ecole Norm. Sup. (3) **69**, 277–302 (1952); these Rev. **14**, 633]. In the present paper it is assumed that the domain is a bicylinder. It is proved that every α-Fuchsian class of sufficiently high dimension contains Fuchsian functions of the same dimension, and that one of these is the sum of a Poincaré series. The proof depends on a construction similar to the one used in the proof of Cousin's first theorem.

H. Tornehave (Copenhagen).

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***Bochner, S. Structure of complex spaces.** Contributions to the theory of Riemann surfaces, pp. 189–201. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

Cet article groupe, d'une part, des résultats obtenus par l'auteur dans de nombreux papiers antérieurs sur les variétés analytiques complexes: métriques kählériennes engendrées par des champs de tenseurs; variétés complexes à singularités algébriques; fonctions automorphes; d'autre part des résultats nouveaux sur les questions suivantes. (1) Séparabilité: le raisonnement de T. Radó [Acta Litt. Sci. Szeged **2**, 101–121 (1925)] prouvant la séparabilité des surfaces de Riemann est repris pour montrer comment il cesse d'être applicable aux variétés analytiques complexes de dimension supérieure à un. (2) Conditions sur les fonctions coordonnées (ou sur un système de champs de tenseurs) suffisantes pour qu'une variété réelle (de dimension réelle paire), munie d'une métrique hermitienne, ait une structure complexe. (3) Invariance de la structure analytique d'une cellule située tout entière dans un voisinage de coordonnées d'une variété analytique complexe. (4) Conjectures sur le problème de l'uniformisation d'une variété analytique complexe. *P. Dolbeault* (Paris).

Loster, C. Sur certaines fonctions homogènes de deux variables complexes. Ann. Soc. Polon. Math. **24** (1951), no. 2, 165–172 (1954).

L'auteur complète des résultats de F. Leja [Ann. Acad. Sci. Tech. Varsovie **3**, 193–206 (1936); Ann. Soc. Polon. Math. **22**, 235–268 (1950); ces Rev. **11**, 653] donnant pour les ensembles fermés bornés de l'espace C^2 des variables complexes x, y une notion métrique construite à partir de la distance $|p_1p_2| = \frac{1}{2}|x_1x_2 - x_2y_1|$ en s'inspirant de la notion de diamètre transfini des ensembles dans C^1. Pour un ensemble e_n de $n+1$ points $(p_0, \cdots, p_n) \in E$ on considère, u étant un point quelconque de C^2, les fonctions:

$$L_n^j(u, e_n) = \prod_k \left|\frac{up_k}{p_jp_k}\right| \quad (j \neq k,\ 0 \leqq k \leqq n)$$

et de même

$$C_n^j(u, e_n) = \prod_k \left|\frac{up_j}{p_kp_j}\right| \quad (R_n^j(u, e_n) = L_n^j(u, e_n)C_n^j(u, e_n))$$

et

$$S_n(u, e_n) = \prod_{j,k} \left|\frac{up_k}{p_kp_k}\right| \quad (j \neq k),$$

puis on forme $L_n(u) = \inf_{e_n \in E}[\max_j L_n^j(u, e_n)]$ et les trois quantités analogues $C_n(u)$, $R_n(u)$, $S_n(u)$. Selon que l'écart $\theta(E)$ de E (construit à partir des quantités $|p_jp_k|$ d'une manière analogue) est fini ou nul, $(L_n(u))^{1/n}$ a une limite $L(u)$ finie positive, ou tend vers $+\infty$ hors de E. Le présent travail démontre l'existence des limites $(C_n(u))^{1/n} \to C(u)$, $(R_n(u))^{1/2n} \to R(u)$, $(S_n(u))^{1/n(n+1)} \to S(u)$ finies ou infinies dans les mêmes conditions et l'égalité $R(u) = S(u)$. Note du référent: $\log L(u)$ et les fonctions analogues s'obtiennent plus naturellement comme lim sup de fonctions plurisousharmoniques $V(x, y)$ satisfaisant à $V(tx, ty) = V(x, y) + \log|t|$ et construites à partir de polynomes homogènes.

P. Lelong (Lille).

Carafa, Mario. Sulle funzioni analitiche di n variabili complesse. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **12** (1953), 267–284 (1954).

D étant un domaine de l'espace $C^n(z_k)$, à frontière γ formée de surfaces analytiques réelles, l'auteur indique, en étendant un exemple de F. Severi [Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **2**, parte 1, 357–411 (1931)] qu'on peut construire sur γ des arcs analytiques C à une dimension réelle tels que pour toute fonction f holomorphe dans D, frontière comprise, la nullité sur C entraîne $f \equiv 0$. La démonstration construit C comme intersection de γ et d'une variété analytique $z_k = \phi_k(z_1)$, $2 \leqq k \leqq n$, en utilisant le fait qu'une fonction holomorphe ϕ_k s'approche de toute valeur au voisinage d'un point singulier essentiel isolé. Un autre exemple intéressant montrant qu'une fonction holomorphe peut être déterminée par ses valeurs sur une variété analytique réelle V_2 à 2 dimensions réelles est donné en considérant dans $|z_k| < \rho$ la variété V_2: $z_1 = \eta be^{i\xi}$, $z_k = \eta g_k(\xi)$, $2 \leqq k \leqq n$, $g_k(\xi)$ étant analytique du paramètre ξ variant sur un segment; f est déterminée par ses valeurs sur V_2 si, pour tout entier $\nu > 0$, on peut choisir les $g_k(\xi)$ de manière que les fonctions $b^{k_1}e^{ik_1\xi}g_2^{k_2}(\xi) \cdots g_n^{k_n}(\xi)$ obtenues pour $k_i > 0$, $\sum k_i = \nu$, soient linéairement indépendantes, ce qui peut être effectivement réalisé. *P. Lelong* (Lille).

Will, Herbert. Zur Funktionentheorie mehrerer Veränderlichen. Über den Satz von Hammerstein. Math. Ann. **127**, 175–180 (1954).

Let G be a region contained in a bounded star-like regularity-region S in the space of n complex variables. Let $\mathfrak{M}$ be a continuous path joining points t_1 and t_2 in the complex t-plane, and assume that there exist mappings A_t: $Z_j = T_j(z_1, \cdots, z_n, t)$ $(j = 1, \cdots, n)$ of S with the following properties. (1) For fixed t in a neighborhood $U(\mathfrak{M})$ the T_j are regular functions in S, and for fixed $z_1, \cdots, z_n$ in S, they are regular functions of t in $U(\mathfrak{M})$. The T_j are continuous in $z_1, \cdots, z_n, t$. (2) If a point P lies in G, then so do all points $A_t(P)$, for $t \in \mathfrak{M}$. (3) For $t = t_1$, A_t is the identity. (4) There exists a hypersphere H lying entirely in G such that for every point P of G the image $A_{t_2}(P)$ lies over H. Under these conditions the author proves that every function which is regular in G can be uniformly approximated in the interior of G by polynomials. The proof is based upon a lemma which states that under the conditions described above, the regularity envelope $\tilde{G}$ of G also satisfies the

assumptions (2) and (4), and in particular, $\tilde{G}$ is schlicht. Now B. Almer [Ark. Mat. Astr. Fys. **17**, no. 7 (1922)] has proved that every regular function in a bounded star-like region R can be uniformly approximated by polynomials in the interior of the regularity envelope of R. Using this result, the lemma mentioned above and a result obtained earlier by the author, the author obtains the proof of his theorem. *W. T. Martin* (Cambridge, Mass.).

***Lelong, P. Fonctions plurisousharmoniques; mesures de Radon associées. Applications aux fonctions analytiques.** Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, pp. 21–40. Georges Thone, Liège; Masson & C^ie, Paris, 1953.

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The larger part of the paper is a review of the theory of plurisubharmonic functions, in particular of the author's contributions [these Rev. **8**, 271; **13**, 932; **14**, 463, 971]. By using exterior differential forms and the theory of distributions (of L. Schwartz), some previous results appear in a more elegant form. A function $V(z)$ of n complex variables z is called "plurisubharmonic" in a domain $D\subset C^n$ if and only if $-\infty\leqq V(z)<\infty$ and, in no subdomain of D, $V(z)\equiv-\infty$ and $V(z)$ is bounded on any compact subset of D and the restriction of $V(z)$ to any 2-dimensional analytic plane intersecting D is subharmonic. Plurisubharmonic (plsh.) functions were introduced by the author and independently by K. Oka, who called them "pseudo-convex" functions [these Rev. **7**, 290]. The author treats (mostly without proofs): an equivalent definition, upper envelope of families of plsh. functions, approximation of a plsh. function by plsh. functions belonging to C^∞, integral mean values with respect to different sets of integration, associated Hermitean and exterior differential forms, special plsh. functions, completions of subclasses of plsh. functions. He introduces the "P-convex domains" (domains convex with respect to plurisubharmonic functions) and states six other properties that are equivalent to the P-convexity. Also, if a domain is pseudo-convex in the sense of E. E. Levi or Behnke and Thullen it is P-convex, and the converse holds if the boundary of the domain is smooth enough. Let D be a domain. Let $L(D)$ be the class of all functions plsh. in D. Let $\lambda_\gamma(D)$ be the smallest class of (plsh.) functions that contains all functions $\log|f|$, f holomorphic in D, and that contains with $V(z)$ also $aV(z)$ $(a>0)$ and with $V_1(z)$ and $V_2(z)$ also $V_1(z)+V_2(z)$ and with a sequence $V_n(z)$ also $\limsup_{z'\to z}\limsup_{n\to\infty} V_n(z')$. This class of functions is (essentially) identical with the class of "Hartogs functions" introduced by Bochner and Martin [these Rev. **10**, 366]. Using the theorem of K. Oka (loc. cit.) that every schlicht and finite pseudo-convex domain is a domain of holomorphy the author shows that $\lambda_\gamma(D)=L(D)$ if D is a domain of holomorphy (a P-convex domain respectively). He shows that conversely the identity $\lambda_\gamma(D)=L(D)$ for all P-convex domains D would have the theorem of Oka as a consequence. He credits the reviewer for having indicated in his thesis [these Rev. **15**, 25] that every function plsh. in a domain D can be continued (as a plsh. function) into the smallest domain of holomorphy containing D. The latter is not true. The reviewer has shown in his thesis: that $\lambda_\gamma(D)=L(D)$ holds for domains of holomorphy and that Oka's theorem would follow if the identity were true for all domains. He also indicated that if $\lambda_\gamma(D)=L(D)$ would hold for all domains then the continuation of plsh. functions mentioned above would follow. Meanwhile the reviewer has shown (unpublished) that there exist domains D and plsh. functions $V(z)$ such that there exists no continuation of $V(z)$ into the smallest domain of holomorphy containing D. Hence we have $\lambda_\gamma(D)\neq L(D)$ for certain domains.

The second part of the paper contains new material. Let V be plsh.; then $d\mu_k=i\pi^{-1}(\partial^2V/\partial z_k\partial\bar z_k)dz_k\wedge d\bar z_k$ is a positive Radon measure. This and related Radon measures are connected with the Riesz decomposition of a plsh. function:

$$V(z)=H(z)-\int\frac{d\mu(\xi)}{[\sum|z_i-\xi_i|^2]^{n-1}}\quad (H(z)\text{ harmonic}).$$

The author then reviews his earlier results on analytic varieties $f=0$ obtained by studying mean values and masses of the special plsh. functions $\log|f|$ (f holomorphic). Utilizing exterior differential forms ("intersection forms" of q plsh. functions), he indicates how these results can be extended to analytic varieties given by more than one holomorphic function $f_1=0,\cdots,f_q=0$.

H. J. Bremermann (Münster).

Thimm, Walter. Untersuchungen über ausgeartete meromorphe Abbildungen. Math. Ann. **127**, 150–161 (1954).

On considère une variété analytique complexe Δ_m, irréductible en 0, définie par des équations $\phi_\nu(x_1,\cdots,x_n)=0$, $1\leqq\nu\leqq N$, et on poursuit l'étude des valeurs limites sur Δ_m de la transformation $A_k(\Delta_m)$ définie par des fonctions $z_j=P_j(x)/g_j(x)$, $1\leqq j\leqq k$, méromorphes en 0; ces limites sont appelées images singulières de $A_k(\Delta_m)$ en 0 comme dans les mémoires précédents [cf. Math. Z. **57**, 456–480 (1953); ces Rev. **14**, 971]. On distingue un cas dit normal, dans lequel, en particulier, les équations $\phi_\nu(x)=0$, $z_jq_j(x)-p_j(x)=0$ admettent la solution $x=0$, quelque soit z, et le cas non normal: dans ce dernier il existe des indices $\lambda_1,\cdots,\lambda_s$ tels que les équations $\phi_\nu=0$, $q_{\lambda_i}=0$, $p_{\lambda_i}=0$ représentent une variété de dimension complexe $\mu\geqq m-a$. Pourqu'on ait le cas normal il faut et il suffit que l'ensemble des images singulières contienne tout l'espace $C^k(z)$. Alors si w est fonction méromorphe en 0, $\not\equiv\infty$, non indéterminée complètement sur Δ_m, et analytiquement dépendante de z zur Δ_m, w est fonction algébrique des z_i sur Δ_m. Ce résultat demeure dans le cas (dit normal généralisé) où les z_i sont liés sur Δ_m par des relations algébriques $G_\lambda(z_j)=0$, définissant une variété irréductible W, l'ensemble des images singulières en 0 de $A_k(\Delta_m)$ étant alors supposé identique à W. Toute image singulière ζ admet dans ce cas un chemin de détermination $\gamma\in\Delta_m$, sur lequel $z=\zeta$ (ζ est dit alors c-accessible) ou est limite de tels points. Pour une représentation méromorphe $A_k(\Delta_m)$, l'adhérence des images singulières en 0 qui sont c-accessibles sur Δ_m est une variété algébrique.

Plusieurs exemples des différents cas étudiés terminent ce travail qu'on rapprochera de deux mémoires précédents [Math. Ann. **125**, 145–164, 264–283 (1952); ces Rev. **15**, 210] dont certains résultats ont été retrouvés par la méthode de la projection analytique [K. Stein, Colloque sur les fonctions de plusieurs variables, Bruxelles, 1953, Thone, Liège, 1953, pp. 97–107; ces Rev. **15**, 695]. *P. Lelong.*

***Bergman, S. Kernel function and extended classes in the theory of functions of complex variables.** Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, pp. 135–157. Georges Thone, Liège; Masson & C^ie, Paris, 1953.

This paper is a summary of some of the author's extensive work in the theory of several complex variables. In chapter I the author explains his approach to several complex variables and stresses in particular the connections with the

theory of partial differential equations and mathematical physics. In chapter II he reviews: the Bergman kernel function, application of orthogonal functions to meromorphic functions (in several variables), the invariant Bergman metric generated by the kernel function, the behavior of the kernel function at the boundary, the theory of representative domains, the connection of the kernel function with various integral minimum problems, distortion theorems for euclidean quantities under pseudo-conformal mappings. This chapter is a very condensed report of extensive material, giving clearly the principal ideas and results without trying to be complete. [Further details (except of the most recent results), also in the form of a review, can be found in S. Bergman, Sur les fonctions orthogonales de plusieurs variables complexes avec applications à la théorie des fonctions analytiques, Gauthier-Villars, Paris, 1947; Sur la fonction-noyau d'un domaine et ses applications dans la théorie des transformations pseudo-conformes, ibid., 1948; The kernel function and conformal mapping, Math. Surveys, no. 5, Amer. Math. Soc., New York, 1950; these Rev. 11, 344; 12, 402.] In chapter III the author reviews the theory of domains with a distinguished boundary surface and applications such as the Bergman-Weil integral formula (that replaces to a certain extent the Cauchy formula in several variables), the "extended class" which enables one to solve boundary-value problems with boundary values prescribed on the distinguished boundary surface, generalized "Green's function" for functions of the extended class, application of the functions of extended class to zero surfaces and pole surfaces of meromorphic functions, functionals connected with the zeros and poles of analytic functions representing a generalization of studies by Nevanlinna and Ahlfors in one variable. The review of these subjects in chapter III of the present paper is more detailed and covers more recent and more extensive material than previous reviews. *H. J. Bremermann* (Münster).

Fourier Series and Generalizations, Integral Transforms

Men'šov, D. E. On convergence in measure of trigonometric series. Amer. Math. Soc. Translation no. 105, 76 pp. (1954).

Translated from Trudy Mat. Inst. Steklov. 32 (1950); these Rev. 12, 254.

Walmsley, Charles. Correction to "Null trigonometric series in differential equations". Canadian J. Math. 6, 447–448 (1954).

See same J. 5, 536–543 (1953); these Rev. 15, 426.

Tsuchikura, Tamotsu. Absolute Cesàro summability of orthogonal series. II. Correction and remark to the previous paper. Tôhoku Math. J. (2) 5, 302–312 (1954).

The author shows that, if $\varphi \in L^p$, $1<p\leqq 2$, φ even and periodic, then a sufficient condition for the summability $|C, \alpha|$, $\alpha>p^{-1}$, of the Fourier series of φ at the origin, is

$$\int_0^\pi \frac{|\varphi(t)|^p}{t}\left|\log\frac{1}{t}\right|^{p-1+\eta} dt<\infty$$

for some $\eta>0$. This condition extends one previously given by the author, viz.,

$$\int_0^t |\varphi(u)|^p du = O\left\{t\left(\log\frac{1}{t}\right)^{-p-\epsilon}\right\}$$

as $t\to+0$, for some $\epsilon>0$ [same J. (2) 5, 52–66 (1953); these Rev. 15, 417]. Either result shows that: (a) if $\varphi\in L^p$, $1<p\leqq 2$, then summability $|C, \alpha|$, $\alpha>p^{-1}$, is a local property of the function [given for $p=1$ by the reviewer, Proc. London Math. Soc. (2) 41, 517–528 (1936)]. S. Yano has shown that: (b) if $\varphi\in L^p$, $p>1$, then the summability $|C, p^{-1}|$ of the Fourier series is not a local property [Tôhoku Math. J. (2) 5, 194–195 (1953); these Rev. 15, 788; the case $p=1$ was given by the reviewer and H. Kestelman, Proc. London Math. Soc. (2) 45, 88–97 (1939)]. The author gives a simple proof for the case $1\leqq p\leqq 2$.

In his previous paper the author stated that (a) also holds for $p>2$, but he now claims that summability $|C, \frac{1}{2}|$ is not a local property, even for a continuous φ. His argument, which employs Rademacher functions, is somewhat intricate and not given in full detail. The reviewer must however point out that it is stated in Math. Rev. that A. Foà [Boll. Un. Mat. Ital. (2) 2, 325–332 (1940); these Rev. 2, 94] obtained both (a) and (b) for all $p>1$. There is therefore a discrepancy in the literature which needs clearing up.

L. S. Bosanquet (London).

Shapiro, Victor L. Circular summability C of double trigonometric series. Trans. Amer. Math. Soc. 76, 223–233 (1954).

Let $T=\sum a_{m,n}e^{i(mx+ny)}$ be a double trigonometric series with circular partial sums

$$S_R(x, y)=\sum_{m^2+n^2\leqq R^2} a_{m,n}e^{i(mx+ny)}.$$

T is said to be circularly convergent to S, at (x_0, y_0), if $S_R(x_0, y_0)\to S$ as $R\to\infty$; it is said to be circularly summable (C, η) to S if $\sigma_R^{(\eta)}(x_0, y_0)\to S$, where

$$\sigma_R^{(\eta)}(x, y)=\frac{\eta}{R^\eta}\int_0^R S_u(x, y)(R-u)^{\eta-1}du.$$

T is summable C at (x_0, y_0) if it is summable (C, η) for some $\eta>0$. The main result is as follows. Let $a_{m,n}=o[m^2+n^2]$, $\gamma\geqq -1$. In order that T should be summable C at (x_0, y_0) to the sum S, it is necessary and sufficient that there exists an integer $r>\gamma+1$ such that the generalized Laplacian $\Delta_r F(x_0, y_0)=s$, where

$$F(x, y)=\frac{a_{0,0}(x+y)^{2r}}{2^r[(2r)!]}+\sum_{m^2+n^2\neq 0}\frac{(-1)^r a_{m,n}}{(m^2+n^2)^r}e^{i(mx+ny)}.$$

The definition of the generalized Laplacian is the following. Let $f(x, y)$ be integrable on the circumferences of all small circles with centre (x_0, y_0). If

$$\frac{1}{2\pi}\int_0^{2\pi} f(x_0+t\cos\vartheta, y_0+t\sin\vartheta)d\vartheta =\alpha_0+\frac{\alpha_1 t^2}{[2!]^2}+\frac{\alpha_2 t^4}{[2^2 2!]^2}+\cdots+\frac{\alpha_r t^{2r}}{[2^r r!]^2}+o(t^{2r})$$

as $t\to+0$, then $\alpha_r=\Delta_r f(x_0, y_0)$. [For circular summability compare S. Bochner, Trans. Amer. Math. Soc. 40, 175–207 (1936).] *W. W. Rogosinski* (Newcastle-upon-Tyne).

Shapiro, Victor L. Generalized Laplacians of the second kind and double trigonometric series. Duke Math. J. 21, 173–178 (1954).

If $f(x, y)$ is integrable near a point (x_0, y_0) and if, for $t>0$,

$$\frac{1}{\pi t^2}\int_0^t \rho d\rho\int_0^{2\pi} f(x_0+\rho\cos\vartheta, y_0+\rho\sin\vartheta)d\vartheta =\alpha_0+\frac{\alpha_1 t^2}{2[2!]^2}+\cdots+\frac{\alpha_r t^{2r}}{(r+1)[2^r r!]^2}+o(t^{2r}),$$

then α_r is called the generalized rth Laplacian of the 2nd kind of f at (x_0, y_0). Let $X=(x, y)$, $M=(m, n)$, $MX=mx+ny$, and $|M|=\{m^2+n^2\}^{1/2}$. If $T=\sum a_M e^{iMX}$ is a double trigonometric series, it is said to be circularly convergent at X if the circular partial sums $S_R(X)=\sum_{|M|\leq R} a_M e^{iMX}$ converge as $R\to\infty$. If, for $\eta>0$, the

$$\sigma_R^{(\eta)}(x)=\frac{2\eta}{R^{2\eta}}\int_0^R S_u(X)(R^2-u^2)^{\eta-1}u\,du$$

converge, then T is circularly summable (C, η) at X. If E is a plane measurable set and $\int_E|S_R(X)-L(X)|^2dX\to 0$ as $R\to\infty$, then $\lim S_R(X)=L(X)$. The following theorem is proved: Let 2α be an integer ≥ 0, r an integer $\geq\alpha+1$, and let $a_M=o(|M|^{2\alpha+1-\epsilon})$, for some $\epsilon>0$. Put

$$F(X)=\frac{a_0(x+y)^{2r}}{2^r[(2r)!]}+\underset{R\to\infty}{\mathrm{l.i.m.}}\sum_{1\leq|M|\leq R}(-1)^r\frac{a_M}{|M|^{2r}}e^{iMX}.$$

If T is circularly summable $(C, 2\alpha)$ at X_0 to the sum s, then the generalized rth Laplacian of the second kind of $F(X)$ exists at X_0 and is equal to s.

In an earlier doctoral dissertation [Univ. of Chicago, 1952, Part III] the author proved a similar theorem concerning Laplacians of the first kind (arising from circumferential means of f). *W. W. Rogosinski.*

Chow, Yuah Shih. On the Cesàro summability of double Fourier series. Tôhoku Math. J. (2) 5, 277–283 (1954).

F. T. Wang proved that, if $\varphi\in L$, φ even and periodic, then its Fourier series at the origin is (i) summable $(C, 1)$ if $\Phi(t)=o\{t/\log(t^{-1})\}$, (ii) summable (C, α), $0<\alpha<1$, if $\Phi(t)=o(t^{1/\alpha})$, where $\Phi(t)=\int_0^t\varphi(u)du$ [J. London Math. Soc. 22, 40–47 (1947); Ann. of Math. (2) 44, 397–400 (1943); these Rev. 9, 182; 4, 272]. The author extends these results to double Fourier series. *L. S. Bosanquet* (London).

Ohkuma, Tadashi. On a certain system of orthogonal step functions. I. Tôhoku Math. J. (2) 5, 166–177 (1953).

A sequence of divisions D_n of the interval $\langle 0, 1\rangle$ is given, with the following properties: (i) if $n>m$, then D_n is a subdivision of D_m; (ii) the maximal length $|D_n|$ of D_n tends to zero. Let E_n denote the number of intervals constituting D_n. The step functions on the intervals of D_n form a linear vector space $\bar{D}_n$. One chooses, in an arbitrary way, an orthonormal basis $\varphi_1, \varphi_2, \cdots, \varphi_{E_1}$ for $\bar{D}_1$ and extends it gradually to all the $\bar{D}_n$. In this way an orthonormal system Ψ of functions φ_k is obtained over $\langle 0, 1\rangle$. The well known systems of Walsh and Haar are particular cases. The main results are as follows. Theorem 1: The partial sums of order E_n of the Fourier series (with respect to Ψ) of a continuous function converge uniformly to $f(x)$. From this follows Theorem 2: The system Ψ is complete in L^2. Estimates for the Fourier coefficient of a continuous function by means of the modulus of continuity are given, and estimates for the Lebesgue constants. The following result is also of interest. If $E_{n+1}<ME_n$, where $M>1$, then the Fourier series of a continuous function is summable $(C, 1)$, with sum $f(x)$, almost everywhere. On the other hand, there exist systems Ψ such that, for some continuous $f(x)$ and some x_0, the Fourier series of f is not summable C (of any order) at x_0.

W. W. Rogosinski (Newcastle-upon-Tyne).

★Achieser, N. I. Vorlesungen über Approximationstheorie. Akademie-Verlag, Berlin, 1953. ix+309 pp. DM 29.00.

A translation of the author's "Lekcii po teorii approksimacii" [Gostehizdat, Moscow-Leningrad, 1947; these Rev. 10, 33].

Kito, Fumiki. On a Fourier-Bessel expansion of special kind. Proc. Fac. Eng. Keio Univ. 5 (1952), 41–44 (1953).

Let $Q(k_s)=J_n(k_sx)G_n'(k_sa)-J_n'(k_sa)G_n(k_sx)$, where the k_s are successive zeros of $J_n'(kb)G_n'(ka)-J_n'(ka)G_n'(kb)$, $b>a>0$, and n is a positive integer. Here $G_n(x)=-\frac{1}{2}\pi Y_n(x)$. It is shown that for an "arbitrary" function $f(x)$ of a real variable $\sum_1^\infty A_sQ(k_s)=\frac{1}{2}\{f(x+0)+f(x-0)\}$, $a<x<b$.

N. D. Kazarinoff (Lafayette, Ind.).

Gel'fand, I. M., and Šilov, G. E. Fourier transforms of rapidly increasing functions and questions of uniqueness of the solution of Cauchy's problem. Uspehi Matem. Nauk (N.S.) 8, no. 6(58), 3–54 (1953). (Russian)

The methods employed by L. Schwartz in his Théorie des distributions [t. I, II, Hermann, Paris, 1950, 1951; these Rev. 12, 31, 833] are here extended to several new function-spaces and to the solution of certain problems in partial differential equations. The basic idea, which goes back to S. L. Sobolev [Mat. Sbornik N.S. 1(43), 39–72 (1936)], is to consider first a certain space Φ of "basic" (complex) functions, with a suitable topology. These functions are defined on R^N. A generalized function is then defined as a continuous linear functional T on Φ. The space of all such functionals is denoted by $T(\Phi)$. The functions in Φ are all infinitely differentiable and behave at infinity in such a way that the Fourier transform

$$\int_{R^N}\exp\{-2\pi i(s_1x_1+\cdots+s_Nx_N)\}\varphi(x)dx=\tilde{\varphi}(s)$$

is defined for all φ in Φ and is again an infinitely differentiable function with good behavior at infinity. $\tilde{\varphi}(s)$ may be defined for certain complex values $s=\{\sigma_1+it_1, \sigma_2+it_2, \cdots, \sigma_N+it_N\}$. The set of all $\tilde{\varphi}$ is denoted by $\tilde{\Phi}$. For $T\in T(\Phi)$, the Fourier transform T is defined as the generalized function on $\tilde{\Phi}$ such that $\tilde{T}(\tilde{\varphi})=T(\varphi_-)$, where $\varphi_-(x)=\varphi(-x)$. For appropriate spaces Φ, every function f which is Lebesgue integrable on compact sets defines a continuous linear functional by $\varphi\to\int_{R^N}f(x)\varphi(x)dx$, and thus a Fourier transform (no longer necessarily a function) is defined for all such functions f, no matter how rapidly they increase as $|x|\to\infty$. Differentiation of generalized functions is defined by the usual formula $(\partial T/\partial x_i)(\varphi)=-T(\partial\varphi/\partial x_i)$. A function f is a multiplier for a space Φ if $\varphi\in\Phi$ implies $f\varphi\in\Phi$ and $\varphi_n\to 0$ in Φ implies $f\varphi_n\to 0$ in Φ.

Before sketching the applications to Cauchy's problem, it is necessary to list some of the spaces Φ and $\tilde{\Phi}$ obtained. The first space S discussed consists of all functions φ which have partial derivatives of all orders such that φ and all partial derivatives of $\varphi\to 0$ as $|x|\to\infty$ more rapidly than any power of $|x|^{-1}$ [see L. Schwartz, loc. cit., t. II, p. 89]. A sequence $\{\varphi_n\}$ of elements of S converges to 0 if and only if for every $\epsilon>0$, natural number r, and mixed partial derivative D^q, $(1+|x|^2)^r|D^q\varphi_n(x)|\leq\epsilon$ for all x and all $n\geq$ some $n_0(r, q, \epsilon)$. The space K consists of all $\varphi\in S$ having compact support [see L. Schwartz, loc. cit., t. I, p. 21]. The space K_p $(p>1)$ consists of all $\varphi\in S$ such that for all D^q, there exist constants C_1 and $C>0$ for which

$$|D^q\varphi(x)|\leq C_1\exp\{-C|x|^p\}.$$

A sequence $\{\varphi_n\}$ in K_p converges to 0 if $\varphi_n\to 0$ uniformly in R^N and $|D^q\varphi_n(x)|\leq C_1\exp\{-C|x|^p\}$, where C and C_1 depend upon q but not on n. The space Z^p $(p\geq 1)$ consists of all $\varphi(x)\in S$ which are extensible to analytic functions of the N complex variables

$$\{z_1, \cdots, z_n\}=\{x_1+iy_1, \cdots, x_N+iy_N\}=x+iy,$$

and such that

$$P[\varphi]=\int_{-\infty+iy}^{\infty+iy}|P(x+iy)\varphi(x+iy)|^2dx<C_1\exp\{C|y|^p\},$$

where P is an arbitrary polynomial and C_1 and C are constants depending upon P and φ. A sequence $\{\varphi_n\}$ in Z^p converges to 0 if $\varphi_n(Z)\to 0$ uniformly on all compact subsets of complex N-space and $P[\varphi_n]\to 0$ for all P and y. The space $Z_p{}^p$ consists of all $\varphi(z_1, \cdots, z_n)$ which are analytic for all values of $z_1, \cdots, z_n$ and such that

$$(*)\qquad |\varphi(z_1, \cdots, z_N)|\leq K\exp\left\{\sum_{j=1}^{N}\epsilon_jC_j|z_j|^p\right\},$$

where the C_j are positive constants and $\epsilon_j=+1$ for z_j non-real and $\epsilon_j=-1$ for z_j real ($j=1, \cdots, N$). Convergence is defined as being uniform on compact sets and with uniform maintenance of a bound (*).

The Fourier transforms of these function-spaces are next computed ($p'=p/(p-1)$): $\tilde{S}=S$; $\tilde{K}_p=Z^{p'}$; $\tilde{Z}^p=K_{p'}$; $\tilde{K}=Z^1$; $\tilde{Z}^1=K$; $\tilde{Z}_p{}^p=\tilde{Z}_{p'}{}^{p'}$. A detailed discussion of Fourier transforms of generalized functions for each of the function spaces is given.

The applications to Cauchy's problem follow the usual procedure. Let $u(x, t)=\{u_1(x, t), \cdots, u_m(x, t)\}$ be a vector function of $x=\{x_1, \cdots, x_N\}$ and the real variable t. Consider the system of differential equations

$$(1)\qquad \frac{\partial u(x,t)}{\partial t}=P\left(\frac{1}{2\pi i}\frac{\partial}{\partial x}, t\right)u(x, t),$$

where P is an m^2-matrix whose elements are linear differential operators of various orders multiplied by continuous functions of t. The initial condition is $u(x, 0)=u_0(x)$. This system may be regarded as a system of equations in generalized vector-functions $T(\varphi)=\{T_1(\varphi), \cdots, T_m(\varphi)\}$, the unknown function u being replaced by an unknown generalized function. By taking the Fourier transform, this system of equations is transformed into the system of ordinary differential equations

$$(2)\qquad \frac{dv(s,t)}{dt}=P(s,t)v(s,t),\quad (v(s,t)=\widetilde{u(x,t)}),$$

where the matrix $P(s, t)$ has elements which are polynomials in S multiplied by continuous functions of t, and the initial condition is $v(s, 0)=v_0(s)=\tilde{u}_0$.

The basic theorems are the following. Let $Q(s, t_0, t)$ be the matrix of the normal fundamental solution of the system (2): $Q(s, t_0, t_0)=E$. I. If the elements of $Q(s, 0, t)$ are multipliers in Φ for all $t\geq 0$, the system (2) has a solution with arbitrary initial generalized vector-function $v_0(s)\in T^{(m)}(\Phi)$. II. If the elements of $Q(s, t, t_0)$ are multipliers in Φ for all t, $0\leq t\leq t_0$, then (2) has a unique solution in the class $T^{(m)}(\Phi)$. For every system (2), let p_0 be the greatest order of the entire functions of s entering in $Q(s, t_0, t)$. III. Then the elements of $Q(s, t_0, t)$ are multipliers in $Z_r{}^r$ for all $r>p_0$. IV. If the vector-function $v_0(x)$ satisfies the inequality

$$|v_0(x)|\leq C_1\exp\{C|x|^{p_0'-\epsilon}\},\quad \epsilon>0,$$

then the system (1) has a solution in generalized vector functions belonging to $T(z_r{}^r)$, where $r=p_0'-\delta$, $\delta>0$. This solution is unique. A number of other theorems are given.

E. Hewitt (Seattle, Wash.).

★Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G. Tables of integral transforms. Vol. I. Based, in part, on notes left by Harry Bateman. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. xx+391 pp. $7.50.

This book is the first volume of two which continue the work begun in "Higher transcendental functions" [v. I, II, McGraw-Hill, New York, 1953; these Rev. 15, 419] within the Bateman Manuscript Project. Definite integrals are tabled. The material is immense: a suitable principle of classification of the integrals had to be found. While Bateman had started to arrange them according to their fields of application, in this volume a considerable part of the material is organized in tables of standard transforms: Fourier, Laplace, Mellin transforms and their inversions; while Hankel transforms are left to the second volume. Thus the reader, in order to compute a given definite integral, has to reduce it to one of these standard forms (the authors state that integrals which cannot be simplified in this way are tabled in the second half of volume II). Obviously integrals may occur under several transforms, e.g. $\int_0^\infty x^{3/2}e^{-ax}J_\nu(bx)dx$ under Laplace, Mellin or Hankel transforms.

Transforms are stated of algebraic functions, arbitrary powers, step functions, etc., of exponential, trigonometric and related functions, of the Gamma, Error and Legendre functions and orthogonal polynomials, of all types of Bessel and related functions, of hypergeometric and of other higher transcendental functions. A table of the notations and definitions of the latter functions and an index of notations are added. The authors concentrate mostly on integrals involving these functions, as there exist a number of good tables, to which they refer in the introduction, and they state a considerable number of new formulae, particularly on Laplace transforms.

Conditions of validity of the formulas are, of course, not given. The sections "General formulas" for Fourier and Mellin transforms (1.1, 2.1, 3.1, 6.1) are far from being complete: for instance, the convolution formula for exponential Fourier transforms is missing, and the Fourier cosine and sine transforms of $x^nf(x)$ are stated only for $n\geq 0$.

This volume contains about 3000 formulas. A very useful book for work both in pure and applied mathematics! The authors have succeeded in systematizing the vast subject.

H. Kober (Birmingham).

Rathie, C. B. A study of a generalisation of the Laplace's integral. Proc. Nat. Acad. Sci. India. Sect. A. 21, 231–249 (1952).

The author computes the transform

$$f(p)=p\int_0^\infty (px)^{m-\frac{1}{2}}e^{-\frac{1}{2}px}W_{k,m}(px)d\alpha(x)$$

for a large number of special cases. *I. I. Hirschman, Jr.*

Varma, R. S. On a generalization of Laplace integral. Proc. Nat. Acad. Sci. India. Sect. A. 20, 209–216 (1951).

The author studies the integral transform

$$f(s)=\int_0^\infty (st)^{m-\frac{1}{2}}e^{-\frac{1}{2}st}W_{k,m}(st)d\alpha(t),$$

and in particular gives several real inversion formulas.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Snehlata. On generalised Laplace transform and self-reciprocal functions. Proc. Nat. Acad. Sci. India. Sect. A. 21, 190–200 (1952).

The author obtains a very complicated formula for $f(s^p)$ under the assumptions

$$f(s)=s^a\int_0^\infty (qst)^{c-\frac{1}{2}}e^{-(p-\frac{1}{2}q)st}W_{k,m}(qst)\phi(t)dt,$$

$$t^\lambda\phi(t)=\int_0^\infty (ty)^{\frac{1}{2}}J_\nu(ty)y^\lambda\phi(y)dy.$$

I. I. Hirschman, Jr. (St. Louis, Mo.).

Snehlata. On generalisations of Laplace Stieltjes transform. I. Proc. Nat. Acad. Sci. India. Sect. A. 21, 51–62 (1952).

Snehlata. On generalisations of Laplace Stieltjes transform. II. Proc. Nat. Acad. Sci. India. Sect. A. 21, 63–73 (1952).

Snehlata. On generalisations of Laplace Stieltjes transform. III. Proc. Nat. Acad. Sci. India. Sect. A. 21, 75–80 (1952).

Snehlata. On generalisations of Stieltjes transform. Proc. Nat. Acad. Sci. India. Sect. A. 21, 180–189 (1952).

If

$$\phi(s)=\int_0^\infty e^{-st}\psi(t)dt \quad\text{and}\quad f(s)=\int_0^\infty e^{-st}\phi(t)dt,$$

then

$$f(s)=\int_0^\infty (s+t)^{-1}\psi(t)dt;$$

that is, the iteration of two Laplace transforms is a Stieltjes transform. The authors consider the case where one or both of the Laplace transforms is replaced by some generalization, the iteration resulting in a generalized Stieltjes transform. The generalization of the Laplace transform employed is

$$g(s)=s^a\int_0^\infty (qsu)^{c-\frac{1}{2}}e^{-(p-\frac{1}{2}q)su}W_{k,m}(qsu)\phi(u)du.$$

The formulas obtained are too complicated to reproduce here. *I. I. Hirschman, Jr.* (St. Louis, Mo.).

Vučković, Vladeta. Die Stieltjes-Transformation die mit der Geschwindigkeit der Exponentialfunktion unendlich klein wird. Srpska Akad. Nauka. Zbornik Radova 35. Mat. Inst. 3, 255–288 (1953). (Serbo-Croatian. German summary)

Let $A(u)$ be of bounded variation on every finite interval, with $A(0)=0$; let

$$S(x)=\int_0^\infty (u+x)^{-1}dA(u),\quad L(s)=\int_0^\infty e^{-su}dA(u).$$

The following are the principal theorems. If

$$(1)\qquad S(x)=O\{\exp(-x^\alpha)\},\quad x\to\infty,\quad 0<\alpha\leqq\tfrac{1}{2},$$

then

$$\chi(s)=\int_0^\infty e^{-su}u^\gamma A(u^{1/\alpha})du,\quad \gamma>-1,$$

is regular at $s=0$. If (1) holds and, with $\beta>-\alpha$, we have (2) $u^\beta[A(v)-A(u)]>-m$ for $u\leqq v\leqq u+u^{1-\alpha}$, then (3) $A(u)=O(u^{-\beta})$, $u\to\infty$. If $L(t)=O\{\exp(-t^{-\alpha/(1-\alpha)})\}$, $t\to0^+$, $0<\alpha\leqq\frac{1}{2}$, and if (2) holds, then (3) holds. If (1) holds with $\alpha>\frac{1}{2}$, then $A(u)\equiv0$. The author gives a historical survey of theorems of the kind he discusses. [Cf. also Pleijel, Mat. Tidsskr. B. 1952, 39–43 (1952); these Rev. 14, 977.]

R. P. Boas, Jr. (Evanston, Ill.).

Söhngen, Heinz. Zur Theorie der endlichen Hilbert-Transformation. Math. Z. 60, 31–51 (1954).

In order to study the finite Hilbert transformation

$$\mathfrak{H}\{F(x)\}=\frac{1}{\pi}\mathrm{P}\int_{-1}^{1}\frac{F(y)}{x-y}dy\quad(-1<x<1)$$

(where the Cauchy Principal Value of the integral is to be taken), the author introduces the transformation

$$\mathfrak{T}_s\{F(x)\}=\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^s F(x)dx.$$

By a change of variables, a connection with the two-sided Laplace transformation is established, and this connection is used to investigate the principal properties of $\mathfrak{T}$. A table of transform pairs, and a table of operations, are given. For any function $F(x)$ in $L^p(-1,1)$, $p>1$, and

$$|\operatorname{Re} s|<1-p^{-1},$$

$$\mathfrak{T}_s\{\mathfrak{H}\{F(x)\}\}=\mathfrak{T}_s\{F(x)\}\cot\pi s-\mathfrak{T}_0\{F(x)\}\operatorname{cosec}\pi s,$$

and this formula is used to investigate the finite Hilbert transformation. It is then proved that for any $G(x)$ in $L^p(-1,1)$, $p>1$, the general solution of the integral equation $\mathfrak{H}\{F(x)\}=G(x)$ for F in L^q is

$$F(x)=-\frac{1}{\pi}\mathrm{P}\int_{-1}^{1}\left(\frac{1-y^2}{1-x^2}\right)^{1/2}\frac{G(y)}{x-y}dy+\frac{c}{\pi(1-x^2)^{1/2}}$$

with an arbitrary constant c: if $p\leqq2$, then F belongs to L^q for all $1<q<p$. The integral equation

$$F(x)+\lambda\mathfrak{H}\{F(x)\}=G(x)$$

is also discussed and solved. For $-1\leqq i\lambda\leqq1$ the solution is uniquely determined, for all other (real or complex) values of λ the general solution contains an arbitrary constant C. The results are briefly applied to a problem in elasticity.

A. Erdélyi (Pasadena, Calif.).

Chak, A. M. Some theorems in operational calculus. I. Ann. Univ. Lyon. Sect. A. (3) 16, 53–62 (1954).

It is known that the Laplace transformation transforms functions which are self-reciprocal in the Fourier sine (cosine) transformation into functions which are self-reciprocal in the sine (cosine) transformation, and kernels transforming classes of functions self-reciprocal in Hankel transformations of different orders are also known [see for instance Titchmarsh, Fourier integrals, Oxford, 1937, sec. 9.14]. Mitra and Bose [Acta Math. 88, 227–240 (1952); these Rev. 14, 555] have rung changes on this theme, and the present paper continues their work, and also extends it from self-reciprocal functions to Hankel transform pairs.

A. Erdélyi (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Geronimus, Ya. L. Polynomials orthogonal on a circle and their applications. Amer. Math. Soc. Translation no. 104, 79 pp. (1954).

Translated from Zapiski Naučno-Issled. Inst. Mat. Meh. Har'kov. Mat. Obšč. (4) 19, 35–120 (1948); these Rev. 12, 176.

Džrbašyan, M. M. Estimates of the derivatives of polynomials and weighted best approximation in a complex region. Akad. Nauk Armyan. SSR. Doklady 17, 3–8 (1953). (Russian. Armenian summary)

The author announces extensions of his previous results [Doklady Akad. Nauk SSSR (N.S.) 84, 5–8, 1123–1126 (1952); these Rev. 13, 841; 14, 164]. He now estimates higher derivatives of polynomials satisfying $|P_n(z)| \leq e^{p(|z|)}$ in an angular region, and states several theorems, involving higher derivatives, on weighted polynomial approximation.

R. P. Boas, Jr. (Evanston, Ill.).

Vidav, Ivan. Sur la solution de H. Pollard du problème d'approximation de S. Bernstein. C. R. Acad. Sci. Paris 238, 1959–1961 (1954).

Bernstein's problem is that of approximating continuous functions, vanishing at $\pm\infty$, by functions $P_n(u)K(u)$, where P_n are polynomials. Pollard's necessary and sufficient conditions on the weight function $K(u)$ involved (1) the existence of a sequence of polynomials $p_n(u)$ such that $p_n(u)K(u) \to 1$ and $\max|p_n(u)K(u)|$ is bounded in n, (2) the condition $\int_{-\infty}^{\infty}(1+u^2)^{-1}\log|K(u)|du = -\infty$ [Proc. Amer. Math. Soc. 4, 869–875 (1953); these Rev. 15, 407]. The author shows that (1) implies (2), except when $K(u)$ is the reciprocal of an entire function.

R. P. Boas, Jr.

Tamadyan, A. P. On a theorem of M. V. Keldyš. Akad. Nauk Armyan. SSR. Izvestiya Fiz.-Mat. Estest. Tehn. Nauki 6, no. 2, 5–11 (1953). (Russian. Armenian summary)

The theorem in question [Mat. Sbornik N.S. 16(58), 1–20 (1945); these Rev. 7, 64] gives a sufficient condition for the possibility of polynomial approximation in a bounded simply-connected region D to analytic functions $f(z)$ such that $\|f\|^2 = \int\int h(z)|f(z)|^2dxdy$ (over D) is finite, approximation being measured in this norm. The author proves the following sharper theorem. Let $z = z(w)$ map $|w| < 1$ on D and suppose that the corresponding approximation is possible in $|w| < 1$ with "weight" $h[z(w)]$. Let $d(z)$ denote the distance from z to the boundary of D. Then a sufficient condition is that $\liminf [d(z)]^3 \log\log\{1/h(z)\} > 0$.

R. P. Boas, Jr. (Evanston, Ill.).

Palamà, Giuseppe. Relazioni tra i polinomi associati alle funzioni di Laguerre e di Hermite. Boll. Un. Mat. Ital. (3) 9, 64–66 (1954).

The author is concerned with the polynomials $G_n(x)$ of degree n satisfying the relation

$$H_n(x)G_n(x) - H_{n+1}(x)G_{n-1}(x) = n!, \quad G_0(x) = 1,$$

where $H_n(x)$ is Hermite's polynomial. In an earlier note [same Boll. (3) 8, 185–193 (1953); these Rev. 15, 123] he has established for the Laguerre function $l_n^{(\alpha)}(x)$ of the second kind the representation

$$(2\pi)^{-\frac{1}{2}}\Gamma(\alpha+1)[L_n^{(\alpha)}(x)I_\alpha(x) + P_n^{(\alpha)}(x)x^{-\frac{1}{2}}e^x],$$

where $I_\alpha(x) = \int_0^x x^{-\alpha-1}e^x dx$; $P_n^{(\alpha)}(x)$ is a certain polynomial of degree $n-1$. Two representations are obtained for $G_n(x)$ involving the latter polynomials (with appropriate values of α) and the Hermite polynomials.

G. Szegő.

Special Functions

Cazenave, René. Calcul des intégrales et des fonctions elliptiques usuelles. Ann. Télécommun. 9, 141–155 (1954).

Continuing an earlier article [same Ann. 9, 103–108 (1954); these Rev. 15, 868] the author describes Landen's and Gauss' transformation, relations with theta functions, the numerical computation of elliptic integrals and elliptic functions from existing tables, and otherwise, relations between complete elliptic integrals and Legendre functions, and the evaluation of certain Bessel function integrals in terms of elliptic integrals.

A. Erdélyi.

***Delsarte, Jean. Sur certains systèmes d'équations aux dérivées partielles à une seule fonction inconnue, et sur une généralisation de la théorie des fonctions de Bessel et des fonctions hypergéométriques.** Premier colloque sur les équations aux dérivées partielles, Louvain, 1953. pp. 35–62. Georges Thone, Liège; Masson & Cie, Paris, 1954.

In section 1 the author proves that a system of n algebraically independent linear homogeneous partial differential equations for a single function of n variables, has a finite number of linearly independent solutions.

In section 2 he introduces a generalized Bessel function and shows that it satisfies a system of partial differential equations. Let C^n [elements $x = (x_1, \cdots, x_n)$] be an n-dimensional vector space, $\bar{C}^n$ [elements $\xi = (\xi_1, \cdots, \xi_n)$] the dual of C^n, and $\langle x, \xi\rangle = \sum x_i\xi_i$ the scalar product. Let $\mathfrak{G}$ be a compact group of linear mappings A of C^n onto itself, and $\mathfrak{G}^*$ the group of the adjoint mappings A^* of $\bar{C}^n$ onto itself. The operation $^\natural$ may be defined on the continuous functions $f(x)$, $f^\natural(x)$ being the average of $f(y)$ over $y = A(x)$ where A runs through $\mathfrak{G}$, and averaging is with respect to the Haar measure of $\mathfrak{G}$. The generalized Bessel function, J, of the first kind of order zero is then obtained by applying $^\natural$ to $\exp\langle x, \xi\rangle$ (with respect to x, ξ, or both). Expansion in a series of homogeneous polynomials, integral relations, and differentiation formulas for this function follow. The partial differential operators which annihilate J are shown to depend on the polynomials in x_i which are invariant under $\mathfrak{G}$, and on the corresponding polynomials in ξ_i. If $n = 2$, $\mathfrak{G}$ is the group of rotations around the origin, then J is $J_0(r)$, the well-known Bessel function.

In section 3 a particular case is investigated in more detail. Here $\mathfrak{G}$ is the group of diagonal substitutions $x_i = a_ix_i$ which leave a given system of p generalized monomials

$$x_1^{m_{j1}}x_2^{m_{j2}}\cdots x_n^{m_{jn}} \quad (j = 1, \cdots, p;\ m_{jk} \text{ integers})$$

invariant. In this case Bessel functions of higher orders are also defined, and most of the formulas can be given more explicitly.

A. Erdélyi (Pasadena, Calif.).

Snehlata. On some infinite integrals involving Struve's functions. Proc. Nat. Acad. Sci. India. Sect. A. 21, 167–173 (1952).

Snehlata. On some infinite integrals involving Bessel functions. Proc. Nat. Acad. Sci. India. Sect. A. 21, 174–179 (1952).

In these two papers the author evaluates integrals such as

$$\int_0^\infty x^{\rho-1}\exp\left(-\frac{c^2x^2}{4}\right)\mathbf{H}_\mu(bx)\phi_\nu(ax)dx,$$

$$\int_0^\infty x^{\rho-1}\phi_\nu(ax)J_\mu(bx)dx,$$

and some similar integrals, where J is the Bessel function, $\mathbf{H}$ the Struve function, and

$$\phi_\nu(x)=\frac{\pi^{1/2}(x/2)^\nu}{\Gamma(\nu+1)}\ {}_1F_1\left(\tfrac{1}{2};\nu+1;-\frac{x^2}{4}\right).$$

The evaluation is purely formal, and involves interchanging the order of integration, or the order of integration and summation. These processes are stated to be "easily justifiable" but their justification imposes certain conditions on the parameters which are either not stated at all, or else stated incorrectly. *A. Erdélyi* (Pasadena, Calif.).

Snehlata. On infinite integrals involving products of Struve's functions. Proc. Nat. Acad. Sci. India. Sect. A. 21, 25–31 (1952).

The author evaluates

$$\int_0^\infty x^{\rho-1}\exp(-ax^2)\prod \mathbf{H}_{\nu_i}(b_i x)dx$$

by expanding each $\mathbf{H}$ in a power series and then integrating term by term. *A. Erdélyi* (Pasadena, Calif.).

Ragab, F. M. Integrals involving *E*-functions and modified Bessel functions of the second kind. Proc. Glasgow Math. Assoc. 2, 52–56 (1954).

Evaluation of the integrals

$$\int_0^\infty \lambda^{m-1}K_n(\lambda)E(p;a_r:q;\rho_s:z/\lambda)d\lambda,$$

$$\int_0^\infty \lambda^{m-1}K_n(\lambda)E(p;a_r:q;\rho_s:\lambda^2 z)d\lambda,$$

and an application of the results to the evaluation of some integrals containing products of Bessel functions. *A. Erdélyi* (Pasadena, Calif.).

Henrici, Peter. Über die Funktionen von Gegenbauer. Arch. Math. 5, 92–98 (1954).

This paper is a continuation of the third part of an earlier paper [Comment. Math. Helv. 27, 235–293 (1954); these Rev. 15, 710]. The curvilinear coordinates used in the present paper are bipolar coordinates, $k=0$ is taken in the wave equation, the expansions obtained are expansions of hypergeometric functions in series of Gegenbauer polynomials, and they resemble the Cayley-Orr identities [see, e.g., Erdélyi et al., Higher transcendental functions, vol. I, McGraw-Hill, New York, 1953, sec. 2.5.2; these Rev. 15, 419]. *A. Erdélyi* (Pasadena, Calif.).

Sips, R. Recherches sur les fonctions de Mathieu. I, II, III, IV, V, VI. Bull. Soc. Roy. Sci. Liège 22, 341–355, 374–387, 444–455, 530–540 (1953); 23, 41–50, 90–103 (1954).

This memoir is motivated by the observation that many known, and some new, properties of Mathieu functions may be deduced from the fundamental formula

$$H_0^{(1)}(kR)=\frac{4i}{\pi}\sum_n\left[\frac{\mathrm{ce}_n(\eta)\,\mathrm{ce}_n(\eta_0)\,\mathrm{Me}_n^{(1)}(\xi)\,\mathrm{Ce}_n(\xi_0)}{\mathrm{ce}_n(0)\,\mathrm{Me}_n^{(1)}(0)}-\frac{\mathrm{se}_n(\eta)\,\mathrm{se}_n(\eta_0)\,\mathrm{Ne}_n^{(1)}(\xi)\,\mathrm{Se}_n(\xi_0)}{\mathrm{se}_n'(0)\,\mathrm{Ne}_n^{(1)}(0)}\right],$$

where ξ, η are elliptic coordinates, $\xi>\xi_0$, and R is the distance between the points (ξ_0,η_0) and (ξ,η). The formula can be interpreted as the decomposition of circular-cylindrical waves into elliptic-cylindrical waves.

After a few introductory remarks on Mathieu functions the author proves this formula which he ascribes to P. M. Morse [Proc. Nat. Acad. Sci. U. S. A. 21, 56–62 (1935)]. He then discusses modified Mathieu functions of the third kind, their integral representations in terms of Mathieu functions, series expansions, etc. Integral equations for Mathieu functions, and integral relations connecting Mathieu functions and associated Mathieu functions are also deduced from the fundamental formula as are expansions of products of Mathieu functions, infinite series involving such products, asymptotic expansions of Mathieu functions for large values of the variable, and the limiting forms of Mathieu functions.

The notation of Mathieu functions in this paper is that used in McLachlan's book [Theory and applications of Mathieu functions, Oxford, 1947; these Rev. 9, 31], except for a different normalization of modified Mathieu functions of the third kind.

Reviewer's remarks. (i) The fundamental formula is equivalent to one of the integral equations satisfied by Mathieu functions, and the great significance of integral equations in the theory of Mathieu functions and similar functions was pointed out by E. T. Whittaker about forty years ago. (ii) The fundamental formula is contained in the so-called addition theorem of Mathieu functions which was given by Schäfke [Math. Z. 58, 436–447 (1953); these Rev. 15, 424] and shown by him to contain most of the important series expansions involving Mathieu functions as particular or limiting cases. *A. Erdélyi* (Pasadena, Calif.).

Piloty, H. Zolotareffsche rationale Funktionen. Z. Angew. Math. Mech. 34, 175–189 (1954).

"Zolotareff functions" are rational analogues of Tchebichef polynomials in the following sense. Tchebichef polynomials provide (among polynomials of a fixed degree, with leading coefficient unity) best uniform approximations of zero in the interval $(-1,1)$, while Zolotareff functions provide, roughly speaking, (among all rational functions of a fixed degree of both numerator and denominator, and with value unity at 1) simultaneously best uniform approximations of zero on $(-1,1)$, and best uniform approximation of infinity on the real axis outside $(-1,1)$. Such functions are used in electrical network theory, and the present paper summarizes their properties.

More precisely, Zolotareff (1877) [cf. Collected works, Izdat. Akad. Nauk SSSR, Leningrad, 1931, 1932, v. 1, p. 372, v. 2, pp. 1–59] determined a_r, $0<a_r<1$, so that

$$F_{2m}(z)=\prod_{\nu=1}^m\frac{z^2-a^2_{2\nu}}{1-a^2_{2\nu}z^2},\quad F_{2m+1}(z)=z\prod_{\nu=1}^m\frac{z^2-a^2_{2\nu-1}}{1-a^2_{2\nu-1}z^2}$$

have the least possible absolute maxima on the interval $(-k^{1/2},k^{1/2})$ where $0<k<1$, and hence the largest possible absolute minima on the real axis outside the interval $(-k^{-1/2},k^{-1/2})$. It turns out that $a_r=k^{1/2}\,\mathrm{sn}\,(rK/n;k)$ where $n=2m$ or $2m+1$, as the case may be, and the notations usual in the theory of Jacobian elliptic functions have been used. An elegant parametric representation is

$$F_{2m}(z)=(-1)^m k_1^{1/2}\,\mathrm{sn}\,(nvK_1;k_1),$$

$$F_{2m+1}(z)=(-1)^m k_1^{1/2}\,\mathrm{sn}\,(nvK_1+K_1;k_1),$$

where $z=k^{1/2}\,\mathrm{sn}\,(vK;k)$, and k_1 is a new modulus for which an explicit formula is derived in the paper, similarly for the new quarter period K_1.

The author describes the behavior of these functions both for real and complex z, determines their maxima on $(-k^{1/3}, k^{1/3})$, and gives formulas for solving the equation $F(z)=A$ in terms of elliptic functions and integrals.
A. Erdélyi (Pasadena, Calif.).

Smith, Jack H., and Storm, Martin L. Generalized off-axis distributions from disk sources of radiation. J. Appl. Phys. **25**, 519–527 (1954).

The authors, in studying the problem of calculating biological radiation doses due to disk sources of radiation, evaluate integrals of the type

$$\int_0^\infty S(\rho)\rho d\rho \int_0^{2\pi} G(R)d\theta, \quad R^2=z^2+\rho^2+\epsilon^2-2\epsilon\rho\cos\theta,$$

by expanding G in powers of $\cos\theta$, ϵ, or ρ prior to integration. Numerical results are given for a typical case.
C. J. Bouwkamp (Eindhoven).

Differential Equations

Sikkema, P. C. Function-theoretic researches on differential operators of infinite order. I, II, III, IV. Nederl. Akad. Wetensch. Proc. Ser. A. **56**, 465–477 (1953); **57**, 176–187, 280–291, 292–305 (1954) = Indagationes Math. **15**, 465–477 (1953); **16**, 176–187, 280–291, 292–305 (1954).

The author continues the investigations of his thesis [Differential operators and differential equations of infinite order with constant coefficients ⋯, Noordhoff, Groningen-Djakarta, 1953; these Rev. **15**, 623]. Results given previously only for functions of growth at most order 1, finite type, are now extended to arbitrary finite order.
R. P. Boas, Jr. (Evanston, Ill.).

da Silva Dias, C. L. Bibliography of theorems of existence, uniqueness, and dependence upon parameters for ordinary differential equations and systems of equations. Bol. Soc. Mat. São Paulo **4** (1949), 31–62 (1951). (Portuguese)

The bibliography is arranged chronologically and spans the interval 1868 to 1950.

Sibuya, Yasutaka. Sur un système des équations différentielles ordinaires non linéaires à coefficients constants ou périodiques. J. Fac. Sci. Univ. Tokyo. Sect. I. **7**, 19–32 (1954).

Sibuya, Yasutaka. Sur un système des équations différentielles ordinaires non linéaires à coefficients constants ou périodiques. II. J. Fac. Sci. Univ. Tokyo. Sect. I. **7**, 107–127 (1954).

Let X, Y, Z denote vectors in complex n-space, and let $(Z;t)_2$ denote a power series in the components of Z with each term of degree $\geqq 2$ and with coefficients that are constant or 2π-periodic in t. Let D denote a square matrix with $\lambda_1, \cdots, \lambda_n$ down the principal diagonal and zeros elsewhere. Consider the vector differential equation (a) $\dot X = DX+(X;t)_2$, where $(X;t)_2$ converges for $\|X\|$ sufficiently small. The object is to reduce (a) by a substitution (b) $X=Y+(Y;t)_2$ to (c) $\dot Y = DY$. It is shown that there exists a unique substitution (b) carrying (a) formally into (c) and meeting a certain further condition. It is also shown that the series $(Y;t)_2$ in this substitution converges if either (i) the n functions $e^{\lambda_j t}$ have a period in common with the coefficients of (a) or (ii) each λ_j is purely imaginary and the solution $X(t)$ of (a) with $X(0)=X_0$ is bounded in t and X_0 uniformly for $t\geqq 0$ and $\|X_0\|$ sufficiently small. A very similar theorem is stated with t complex.

In the second paper t is real and not all λ_j are purely imaginary. If the solution of (c) is Y, with components $y_j=C_je^{\lambda_j t}$, then a formal solution of (a) is $X=Y+(C,t)_2$. Convergence is obtained by setting selected C's equal to zero, for example, so as to guarantee both that nonvanishing y_j all have λ_j with real part $\geqq 0$ (or $\leqq 0$) and that those y_j with purely imaginary λ_j have a period in common with the coefficients of (a).
F. A. Ficken (Knoxville, Tenn.).

Hukuhara, Masuo. Sur les équations différentielles linéaires à coefficients périodiques et contenant un paramètre. J. Fac. Sci. Univ. Tokyo. Sect. I. **7**, 69–85 (1954).

In the vector differential equation $\dot X = A(t,\epsilon)X$, suppose that the coefficients $a_{jk}(t,\epsilon)$ are continuous in (t,ϵ) and holomorphic in ϵ for $-\infty<t<\infty$ and $|\epsilon|<\rho_0$ and are ω-periodic in t. The equation $\dot X = A(t,0)X$ can be reduced by a linear transformation with periodic coefficients to $\dot Y = AY$ with A constant and in Jordan normal form. In this paper the equation $\dot X = A(t,\epsilon)X$ is reduced by linear transformations involving power series in ϵ with periodic coefficients to $\dot Y = A(\epsilon)Y$ with $A(\epsilon)$ in normal form. Necessary and sufficient conditions are given for the convergence of the power series for sufficiently small ϵ. An important role in the argument is played by triangular matrices of the form $\sum_0^{n-1} a_k E^k$, where E has elements $e_{i,i+1}=1$ and $e_{ij}=0$ otherwise; such matrices are here called "obliquely constant".
F. A. Ficken (Knoxville, Tenn.).

Demidovič, B. P. On a case of almost periodicity of a solution of an ordinary differential equation of 1st order. Uspehi Matem. Nauk (N.S.) **8**, no. 6(58), 103–106 (1953). (Russian)

The author proves the following theorem: If $f(x)$ is a monotonic function with a continuous derivative, and $g(t)$ is an ordinary almost periodic function with a bounded indefinite integral, then every bounded solution of the differential equation $dx/dt=f(x)+g(t)$ is an almost periodic function.
H. Tornehave (Copenhagen).

Feščenko, S. F. Estimate of the error in the asymptotic behavior of integrals of ordinary linear differential equations having a parameter. Dopovidi Akad. Nauk Ukrain. RSR **1951**, 156–162 (1951). (Ukrainian. Russian summary)

The author considers a system of an n-vector and $n\times n$ matrix

$$\dot x = A(\tau,\epsilon)x, \quad \tau=\epsilon t, \quad |\epsilon|<1$$

under the assumptions that

$$A(\tau,\epsilon)=\sum_{s=0}^{\infty}\epsilon^s A^s(\tau)$$

and that the characteristic roots $\lambda_1(\tau), \cdots, \lambda_n(\tau)$ of $A^0(\tau)$ are such that $\lambda_i\neq\lambda_j$, $i\leqq r$, $j>r$. In a previous paper [same Dopovidi **1949**, no. 1] a certain formal solution of the system had been obtained. This solution is now shown to have asymptotic character and an estimate of the error of the mth approximation is given [additional reference: N. N. Bogoliubov, On some statistical methods in mathematical physics, Izdat. Akad. Nauk Ukrain. SSR, 1945; these Rev. **8**, 37].
S. Lefschetz (Princeton, N. J.).

Lewis, D. C. Differential equations referred to a variable metric. Amer. J. Math. **73**, 48–58 (1951).

In this paper the author sharpens results he obtained in an earlier paper [same J. **71**, 294–312 (1949); these Rev. **10**, 708]. By introducing a metric which is a function of t, the independent variable of a differential system, he obtains theorems on the boundedness of solutions of the differential system which specialize appropriately in the case of a linear system. *W. Leighton* (Pittsburgh, Pa.).

Staševskaya, V. V. On inverse problems of spectral analysis for a class of differential equations. Doklady Akad. Nauk SSSR (N.S.) **93**, 409–411 (1953). (Russian)

The inverse problem for $y''+[\lambda-q(x)]y=0$ on $(0, \infty)$ with $q(x)$ real and integrable on $[0, b]$ for any $b<\infty$ has been considered by Gelfand and Levitan [Izvestia Akad. Nauk SSSR. Ser. Mat. **15**, 309–360 (1951); these Rev. **13**, 558]. Here the inverse problem for

$$y''+[\lambda-q(x)-n(n-1)/x^2]y=0$$

is considered where for any $\epsilon>0$ and any a

$$\int_0^a |xq(x)|^{2+\epsilon}dx<\infty$$

and similar results to those of Gelfand and Levitan are obtained. *N. Levinson* (Cambridge, Mass.).

Germeier, Yu. B., and Irger, D. S. On approximate representations of solutions of linear differential equations of second order. Doklady Akad. Nauk SSSR (N.S.) **93**, 961–964 (1953). (Russian)

Let $p(t)$ and $q(t)$ be continuous functions for $a\leqq t<b$ $(b\leqq\infty)$ and consider (*) $\ddot{x}+2p(t)\dot{x}+q(t)x=0$ on $[a, b)$. Let $\omega_j(t)$, $j=1, 2$, be continuous on $[a, b)$. Then $\exp\int_a^t\omega_j(s)ds$ are said to be approximate representations of solutions of (*) if there exist continuous $y_j(t)$ $(j=1, 2)$ on $[a, b)$ tending to non-vanishing limits as $t\to b$ and such that $y_j(t)\exp\int_a^t\omega_j(s)ds$ are solutions of (*). Let $R[\omega(t)]=\dot{\omega}+\omega^2+2p\omega+q$ and $\rho(t)=\max|R[\omega_j(t)]|/|\omega_2(t)-\omega_1(t)|$. Let

$$\Omega(t)=\omega_2(t)-\omega_1(t)\neq 0.$$

Let $M[\Omega]=\sup\exp\int_{t_1}^{t_2}\operatorname{Re}\Omega(s)ds$ for $a\leqq t_1\leqq t_2<b$. Let $\int_a^b\rho(s)ds<\infty$ and let at least one of the quantities $M[\Omega]$, $M[-\Omega]$ be finite. Then $\exp\int_a^t\omega_j(s)ds$ are approximate representations of solutions of (*). Further results and many applications are given showing how this theorem includes many known results. *N. Levinson* (Cambridge, Mass.).

★Malkin, I. G. Teoriya ustoĭčivosti dviženiya. [Theory of stability of movement.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 432 pp. 17 rubles.

This is a clear and systematic exposition of the theory of stability, including the most important results achieved in this field up to the recent years. The first three chapters are rather elementary in character and include clearly stated and proved theorems and examples of applications to typical physical problems, so that engineers and technicians with a moderate mathematical training shall be able to use them in the analysis of the majority of problems appearing in practice. The following chapters are of a deeper mathematical content and require a greater maturity to master them. A short account of the contents follows: Chapter I: introduction, definitions, examples. Chapter II: the "second method" of Lyapunov, fundamental theorems on (simple) stability, asymptotic stability and unstability and their application to autonomous systems. Chapter III: stability in the first approximation in the non-critical cases. Chapter IV: critical cases for autonomous systems with one vanishing or two purely imaginary characteristic roots; the problem of centers. Chapter V: general theorems of Lyapunov and Četaev for non-autonomous systems, linear systems with periodic coefficients, reducibility, methods for the approximate calculation of the characteristic exponents, stability in the first approximation for periodic systems in the non-critical and in the simplest critical cases. Chapter VI: general theorems of the author on stability under permanently acting perturbations, theorems of the author, Persidskiĭ and the reviewer on the reciprocal of Lyapunov's theorem on asymptotic stability, applications to stability under permanently acting perturbations; general linear systems, characteristic numbers of Lyapunov, regular systems, theorems of Lyapunov, Perron, Persidskiĭ, Četaev, Poincaré and the author about stability and other properties of characteristic numbers; general theorems on stability in the first approximation; critical cases of autonomous and periodic systems with two vanishing, or one vanishing and a pair of purely imaginary, or two pairs of purely imaginary characteristic exponents. *J. L. Massera* (Montevideo).

Malkin, I. G. On the reversibility of Lyapunov's theorem on asymptotic stability. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 129–138 (1954). (Russian)

Consider the n-vector equation (euclidean metric)

$$\dot{x}=X(t;x), \tag{1}$$

in the region $t>0$, $\|x\|\leqq H$, with $X(t;0)=0$. Lyapunov has shown that the origin is asymptotically stable if there exists a definite positive function (in his sense) $V(t;x)$ whose time derivative $V'(t;x)$ is definite negative, and such, moreover, that $V\to 0$ with x uniformly in t. The author proves essentially the converse of this proposition. He had already dealt with this problem for $n=2$ [Malkin, Sbornik Naučnyh Trudov Kazanskogo Aviacionnogo Inst. no. 7 (1937)]; Barbashin [Doklady Akad. Nauk SSSR (N.S.) **72**, 445–447 (1950); these Rev. **12**, 182] had settled the general autonomous case, and Massera [Ann. of Math. (2) **50**, 705–721 (1949); these Rev. **11**, 721] the general autonomous periodic case.

Let $x=F(t;x^0;t_0)$ be the solution of (1) through $(t_0;x^0)$. Then: Theorem 1. If there exists a $\delta>0$ such that $\|F\|\to 0$ as $t\to+\infty$ for $\|x^0\|\leqq\delta$, $t^0\geqq 0$ uniformly in $(t_0;x^0)$, then there exists a $V(t;x)$ for which the Lyapunov property holds. Theorem 2. If $V(t;x)$ à la Lyapunov exists then there exists a suitably small δ such that, for t^0, $x_0\in\{t\geqq 0;\|x\|\leqq\delta\}$, $F(t;x^0;t_0)\to 0$ uniformly in $(x^0;t_0)$. The assumption is made throughout that in the region considered X is continuous and of class C^1 in x. [Additional references: Malkin, Akad. Nauk SSSR. Prikl. Mat. Meh., **8**, 241–245 (1944); these Rev. **7**, 298; see also the book reviewed above; Gorshin, Izvestiya Akad. Nauk Kazah. SSR **1948**, no. 56, Ser. Mat. Meh. **2**, 46–73 (1948); these Rev. **14**, 48]. *S. Lefschetz.*

Krasovskiĭ, N. N. On stability of motion in the large for constantly acting disturbances. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 95–102 (1954). (Russian)

Consider the n-vector equation

$$\dot{x}=X(x),\quad X(0)=0 \tag{1}$$

where X is continuous and the associated system

$$\dot{x}=X(x)+R(x;t), \tag{2}$$

where generally $R(0;t)\neq 0$. The origin is said to be stable in

the large for constantly acting disturbances (=s.l.c.d.) if it is stable in the small for constantly acting disturbances [see the book reviewed second above] and furthermore the following holds: Let $\|x\|=\sup|x_i|$, and similarly for other vectors. Then, given ϵ, there exists η such that if R is such that outside $\|x\|<\epsilon$, $\|R\|<\eta f(r)$, $r=(\sum x_i^2)^{1/2}$, then every solution of (2) has the property that $\lim\sup\|x(t)\|\to 0$ as $t\to+\infty$. Here $f(r)$ is a given function characterizing the admissible growth of R.

The object of the present paper is to give sufficient condition for s.l.c.d. for certain systems considered in various papers [Erugin, these Rev. 14, 376; Krasovskiĭ, ibid. 14, 376; Malkin, ibid. 14, 48; Eršov, ibid. 14, 752]. The assumption is made throughout that $f(r)=r$, and x, y are now the plane cartesian coordinates. Consider the system

$$(3)\qquad \dot{x}=F(x,y),\quad \dot{y}=f(z),\quad z=ax-by,$$

where a, b are constants with $a\neq 0$, F and f are continuous, $F(0,0)=f(0)=0$, and F, f satisfy the condition for uniqueness of solutions at the origin. Then: Theorem. If the solutions of the equation

$$\begin{vmatrix} F_x-\lambda & F_y \\ f_x & f_y-\lambda \end{vmatrix}=0$$

have their real parts $<-\delta$, where δ is a suitably small positive number, and this for all x, y, then the system (3) is s.l.c.d.

Consider now the system

$$(4)\qquad \dot{x}=f_1(x)+ay,\quad \dot{y}=f_2(x)+by,$$

where a, b are constants with $a\neq 0$, the f_i are continuous with $f_i(0)=0$, and $|f_i(x)|<M|x|$, where M is a suitably large constant. Then the same theorem holds relative to the roots of

$$\begin{vmatrix} x^{-1}f_1(x)-\lambda & a \\ x^{-1}f_2(x) & b-\lambda \end{vmatrix}=0.$$

Finally, the same theorem holds regarding the system

$$(5)\qquad \dot{x}=f_1(x)+ay,\quad \dot{y}=bx+f_2(y),\quad |f_i(s)|<M|s|,$$

as regards the roots of

$$\begin{vmatrix} x^{-1}f_1(x)-\lambda & a \\ b & y^{-1}f_2(y)-\lambda \end{vmatrix}=0.$$

[Additional references: Barbašin and Krasovskiĭ, these Rev. 14, 646; Barbašin, ibid. 14, 376; Krasovskiĭ, ibid. 14, 1087.]

S. Lefschetz (Princeton, N. J.).

Krein, M. G. On inverse problems of the theory of filters and λ-zones of stability. Doklady Akad. Nauk SSSR (N.S.) 93, 767–770 (1953). (Russian)

Let $M(x)$ be a nondecreasing function on $(-\infty,\infty)$, $M(x)=M(x-0)$, and let there exist a $T>0$ such that $M(x+T)=M(x)+M(T)$. Let λ be a parameter and consider the equation

$$(*)\qquad y(x)=y(0)+y'(-0)x-\lambda\int_0^x(x-s)y(s)dM(s).$$

(If M is absolutely continuous and $\rho=M'$ then (*) is equivalent to $y''+\lambda\rho(x)y=0$ where ρ has period T.) Let $\phi(x;\lambda)$ and $\Psi(x;\lambda)$ be the solutions of (*) satisfying $y(0)=1$, $y'(-0)=0$ and $y(0)=0$, $y'(-0)=1$ respectively. Let $A(\lambda)=(\phi(T;\lambda)+\Psi'(T-0;\lambda))/2$. Then A is called the Λ-function associated with the mass distribution M. Necessary and sufficient that A be a Λ-function associated with some M is (1) $A(\lambda)=\prod(1-\lambda/a_j)$, $0<a_1<a_2<\cdots$ and $\sum(1/a_j)<\infty$, and (2) all roots of $A^2(\lambda)-1=0$ are nonnegative. Next let $\mu_1<\mu_2<\cdots$ be the simple roots of $A^2(\lambda)-1=0$. Let $\mu_0=0$ and let

$$R(\lambda)=\lambda\prod(1-\lambda/\mu_j),\quad \sum(1/\mu_j)<\infty.$$

Then R is called the filter function associated with M. Necessary and sufficient that an $R(\lambda)$ of the above form be a filter function for some M is that there exist entire functions (A,B) of genus zero and with positive roots which satisfy $A^2(\lambda)+R(\lambda)B^2(\lambda)=1$. Other results are given.

N. Levinson (Cambridge, Mass.).

Wasow, Wolfgang. Asymptotic solution of the differential equation of hydrodynamic stability in a domain containing a transition point. Ann. of Math. (2) 58, 222–252 (1953).

In the theory of hydrodynamic stability it is of importance to know the asymptotic behavior for large λ of solutions of equations of the form

$$(1)\qquad u^{(4)}+\sum_{j=1}^{4}a_j(x)u^{(4-j)}+\lambda^2\sum_{j=0}^{2}b_j(x)u^{(2-j)}=0$$

in the neighborhood of a point where $b_0(x)$ has a simple zero. In the case of real x, real coefficients, and pure imaginary λ^2, Tollmien [Z. Angew. Math. Mech. 25/27, 33–50, 70–83 (1947); these Rev. 9, 476] has developed a complete asymptotic theory. However, for the physical applications it is necessary to find a fundamental system of solutions of (1) with known asymptotic behavior in a full complex neighborhood of the zero of $b_0(x)$. In this paper the author constructs such a fundamental system. He limits himself, as in the hydrodynamical problem, to the case that $b_1(x)\equiv 0$, although he considers this restriction inessential. The method of the paper is to show that the asymptotic behavior of the given equation resembles in all essential features that of the equation,

$$(2)\qquad y^{(4)}+\lambda^2(zy^{(2)}+y)=0,$$

which the author has previously studied in detail [Ann. of Math. (2) 52, 350–361 (1950); these Rev. 12, 261]. For equation (2) a fundamental system of solutions can be constructed explicitly whose asymptotic behavior is known in a full neighborhood of $z=0$. From these solutions the author constructs fundamental solutions of (1) which are asymptotically equal to the former, this fact being proved by an appropriate comparison of (1) and (2). For applications of these results see the author's paper [J. Research Nat. Bur. Standards 51, 195–202 (1953); these Rev. 15, 573].

D. Gilbarg (Stanford, Calif.).

Sternberg, H. M., and Sternberg, R. L. A two-point boundary problem for ordinary self-adjoint differential equations of fourth order. Canadian J. Math. 6, 416–419 (1954).

The authors consider the two-point homogeneous vector boundary problem

$$[P_0(x)u'']''-[P_1(x)u']'+P_2(x)u=0,$$
$$u(x_1)=u'(x_1)=u(x_2)=u'(x_2)=0,$$

where the $P(x)$ are real $m\times m$ symmetric matrix functions with $P_0(x)$ positive definite and $P_i(x)$ of class C^{2-i} on $[a,\infty)$. A solution is a real m-dimensional column vector of class C^2, such that $P_i(x)u^{(2-i)}$ is of class C^{2-i}, which satisfies the system. The authors prove that, if each of the matrices $P_0(x)-k_0I$, $k_0>0$, $P_1(x)$ and $P_2(x)$ is negative semidefinite in $[a,\infty)$ and if there exists an $a_0\geq a$ such that for

$a_0<x_1<x_2<\infty$ the only solution of the system is the zero vector, then $P_1(x)$, $x^2P_2(x)\in L(a,\infty)$ and for each constant unit m-vector π, the quadratic forms $x\pi^*\int_x^\infty P_1(t)dt\pi$ and $x\pi^*\int_x^\infty t^2P_2(t)dt\pi$ stay bounded as $x\to+\infty$. The proof is obtained by reduction to a problem considered by R. L. Sternberg [Duke Math. J. 19, 311–322 (1952); these Rev. 14, 50] and the result forms a partial converse of a theorem previously proved by Kaufman and R. L. Sternberg [ibid. 20, 527–531 (1953); these Rev. 15, 530]. *E. Hille.*

★Lewin, W. I., und Grosberg, J. I. Differentialgleichungen der mathematischen Physik. Verlag Technik, Berlin, 1952. 484 pp.

A translation of the authors' "Differencial'nye uravneniya matematičeskoĭ fiziki" [Gostehizdat, Moscow-Leningrad, 1951; these Rev. 13, 42].

Hornich, H. Lösung der verallgemeinerten Eulerschen Differentialgleichung für homogene Funktionen. Ann. Mat. Pura Appl. (4) 36, 361–365 (1954).

Soit l'équation aux dérivées partielles

$$(*)\qquad \sum_{i,k=1}^{n} a_{ik}x_i\frac{\partial u}{\partial x_k}=\sum f_{l_1\cdots l_n}x_1^{l_1}\cdots x_n^{l_n},$$

où $a_{ik}=a_{ki}=$const, dtm $|a_{ik}|\neq 0$, la série entière donnée au second membre étant supposée convergente au voisinage de l'origine. Cette équation n'admet de solution régulière, i.e., développable en série de puissances entières des variables au voisinage de l'origine, que pour des a_{ik} particuliers [cf. Hornich, Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 11, 125–133 (1952); ces Rev. 15, 626]. L'auteur montre qu'une transformation linéaire convenable des variables $x_1,\cdots,x_n$ ramène (*) à une forme normale pour laquelle il y a toujours une solution développable en série de puissances entières de $x_1,\cdots,x_{n-1}$, mais non nécessairement entières de x_n. *H. G. Garnir* (Liège).

Diaz, Joaquin, et Ludford, Geoffrey. Sur la solution des équations linéaires aux dérivées partielles par des intégrales définies. C. R. Acad. Sci. Paris 238, 1963–1964 (1954).

Given the equation

$$(1)\qquad L(u)\equiv u_{xy}+a(x,y)u_x+b(x,y)u_y+c(x,y)u=0,$$

if $E(x,y,t)$ is a solution of

$$(1-t^2)(E_{yt}+aE_t)-t^{-1}(E_y+aE)+2xtL(E)=0$$

which satisfies certain conditions at $t=0$, $t=\pm1$, then, according to S. Bergman, the expression

$$u(x,y)=\int_{-1}^{+1}E(x,y,t)f(\tfrac{1}{2}x(1-t^2))\frac{dt}{(1-t^2)^{1/2}}$$

is a solution of (1). Now, the authors point out that this is equivalent to a statement made by Le Roux [Ann. Sci. Ecole Norm. Sup. (3) 12, 227–316 (1895)] to the effect that if $U(x,y;\alpha)$ is a family of solutions of (1) then so is

$$u(x,y)=\int_{\alpha_0}^{x}U(x,y;\alpha)f(\alpha)d\alpha\quad(\alpha_0=\text{constant})$$

provided U satisfies $U_y+a(x,y)U=0$ along the characteristics $x=\alpha$ of (1). *S. Bochner* (Princeton, N. J.).

Szarski, J. Evaluation du domaine de régularité du conoïde caractéristique. Ann. Soc. Polon. Math. 24 (1951), no. 2, 85–110 (1954).

This paper is concerned with the direct, characteristic conoid issuing from the origin corresponding to a linear, hyperbolic partial differential equation of second order

$$\sum_{i,k=1}^{m} a_{ik}(x_1,\cdots,x_m)\frac{\partial^2 u}{\partial x_i\partial x_k}=0\quad(a_{ik}=a_{ki})$$

with coefficients which are at least of class C^2 within a sphere $\sum_{i=1}^m x_i^2\leq R^2$. Most of the discussion is concerned with the special case in which $a_{im}=0$ $(i=1,\cdots,m-1)$, x_m being the time-like variable, and is based upon estimates which are obtained for the solutions of the bicharacteristic equations and for certain of their derivatives. It is seen in this case that the conoid will be of class C', and can be represented as $x_m=\phi(x_1,\cdots,x_{m-1})$, in a neighborhood $S:\sum_{i=1}^{m-1}x_i^2<\delta^2$ of 0 with a radius δ which depends upon the following quantities: m, R, bounds for the a_{ik} and for their first and second partial derivatives, and bounds for the eigenvalues of the characteristic quadratic form $\sum_{i,k=1}^{m-1}a_{ik}\lambda_i\lambda_k$. A lower bound for the altitude of the conoid in the neighborhood S also is determined by these quantities. The general case is reduced to the special case by a suitable change of coordinates, and analogous results are obtained. For this purpose, the a_{ij} are assumed to be of class C^3, however, and the ensuing estimates depend upon bounds for the third derivatives of these coefficients in addition to the previously mentioned quantities. *A. Douglis* (New York, N. Y.).

Olevskiĭ, M. N. On the equation

$$A_p u(P,t)=\left(\frac{\partial^2}{\partial t^2}+p(t)\frac{\partial}{\partial t}+q(t)\right)u(P,t)$$

(A_p a linear operator) and the solution of Cauchy's problem for a generalized Euler-Darboux equation. Doklady Akad. Nauk SSSR (N.S.) 93, 975–978 (1953). (Russian)

Let $p_1(t)$, $p_2(t)$, $q_1(t)$ and $q_2(t)$ be arbitrary continuous functions in a given interval $(t\geq 0)$ with continuous first derivatives. $v_1(x_1,\cdots,x_m;t)$ and $v_2(x_1,\cdots,x_m;t)$ (or $v_1(P,t)$ and $v_2(P,t)$) are solutions of the differential equation

$$A_p v=\frac{\partial^2 v}{\partial t^2}+p_2(t)\frac{\partial v}{\partial t}+q_2(t)v$$

with the initial conditions

$$v_1(P,t)|_{t=0}=f(P),\quad \left.\frac{\partial v_1}{\partial t}\right|_{t=0}=0,$$

$$v_2(P,t)|_{t=0}=0,\quad \left.\frac{\partial v_2}{\partial t}\right|_{t=0}=f(P).$$

The linear operator A_p on the left-hand side is supposed to be

$$A_p=\sum_{i,k=1}^{m}a_{ik}(P)\frac{\partial^2}{\partial x_i\partial x_k}+\sum_{i=1}^{m}b_i(P)\frac{\partial}{\partial x_i}+c(P).$$

Then the solutions $u_1(x_1,\cdots,x_m;t)$ and $u_2(x_1,\cdots,x_m;t)$ of the differential equation

$$A_p u=\frac{\partial^2 u}{\partial t^2}+p_1(t)\frac{\partial u}{\partial t}+q_1(t)u$$

can be expressed by the relations

$$u_i(P,t)=r(t)v_i(P,t)+\int_0^t K_i(\xi,t)v_i(P,\xi)d\xi,$$

$$r(t)=\exp\left(\frac{1}{2}\int_0^t(p_2-p_1)dt\right),\quad i=1,2.$$

Their initial conditions are the same. The functions $K_i(\xi,t)$ are solutions of a hyperbolic differential equation. There

follows a discussion of special cases (e.g. $p_1=p_2$, $q_1=q_2$). Moreover, the author deals with the problem of Cauchy in the case of several generalizations of the Euler-Darboux differential equation. The type of these differential equations is

$$A_\beta u = u_{tt} + al^\beta \cot l^\beta\, tu_t + bu.$$

Some results of Gelfand and Courant are special cases of the theory given by the author. *M. Pinl* (Cologne).

Takahashi, Takehito. Invariant delta functions in the sense of distributions. Progress Theoret. Physics 11, 1–10 (1954).

L'auteur effectue, sans donner toutes les démonstrations, certains calculs connus (mais ambigus) de la mécanique quantique, en se plaçant au point de vue rigoureux des distributions. Notations: espace-temps des

$$(x,t),\ x=(x_1,x_2,x_3);\quad x^2=\sum_{j=1,2,3} x_j^2;\quad dx=dx_1dx_2dx_3;$$

$$-\Box=\partial^2/\partial t^2-\partial^2/\partial x_1^2-\partial^2/\partial x_2^2-\partial^2/\partial x_3^2;\quad \sigma^2=t^2-x^2;$$

$s=(t^2-x^2)^{1/2}$ pour $t^2-x^2\geqq 0$, 0 pour $t^2-x^2\leqq 0$; $s^+=s$ pour $t\geqq 0$, 0 pour $t<0$.

(1) L'auteur déduit la distribution de Green G (pour l'avenir) de l'opérateur différentiel $-(\Box+\kappa^2)$, à partir de la solution classique (donnée par Poisson pour $\kappa=0$) du problème de Cauchy sans second membre avec les données initiales $u(x,0)=0$, $\partial u(x,0)/\partial t=f(x)$. Cette distribution, notée Δ^R en mécanique quantique, est bien connue: c'est une distribution en x dépendant continuement du paramètre t: $\Delta^R=D^R-(\kappa/4\pi s^+)J_1(\kappa s^+)$, où D^R, valeur de Δ^R pour $\kappa=0$, vaut 0 pour $t\leqq 0$, et $t\mu_t$ pour $t\geqq 0$, μ_t étant la masse $+1$ répartie de façon homogène sur la sphère $x^2=t^2$ de l'espace R^3.

(2) L'auteur effectue un calcul de convolution

$$\Delta^{(1)}=(1/\pi)\Delta^*P(t^{-1}),$$

$P(t^{-1})$ étant la distribution $\delta_x\times vp(t^{-1})$. On trouve $\Delta^{(1)}=D^{(1)}+$ une fonction de σ, faisant intervenir les fonctions de Neumann N_1 et de Kelvin K_1, ayant une singularité logarithmique sur la surface du cône de lumière:

$$(\kappa/4\pi\sigma)N_1(\kappa\sigma)+(1/2\pi^2\sigma^2)\text{ pour }\sigma^2>0,$$
$$(\kappa/2\pi^2(-\sigma^2)^{1/2})K_1(\kappa(-\sigma^2)^{1/2})-(1/2\pi^2\sigma^2)\text{ pour }\sigma^2<0.$$

(3) Si $F(t)$ est une fonction qui se développe pour $t\to 0^+$ suivant:

$$F(t)=\sum_{m_0\geqq m>0} t^{-m}(a_m+b_m\log t)+a+b\log t+o(1),$$

on pose $\mathrm{Pf}_{t\to 0}F(t)=a$, $\mathrm{Pl}_{t\to 0}F(t)=b$. Appelant alors V_ϵ^+ le volume $(1-\epsilon)^2t^2-x^2\geqq 0$ $(\epsilon>0)$, et V_ϵ^- le volume

$$t^2-(1-\epsilon)^2x^2\leqq 0,$$

l'auteur définit les distributions:

$$\mathrm{Pf}\ (\text{resp. Pl})s^{-2}(\varphi)$$
$$=\mathop{\mathrm{Pf}}_{\epsilon\to 0}\ (\text{resp. }\mathop{\mathrm{Pl}}_{\epsilon\to 0})\left[\lim_{\eta\to 0}\iiiint_{V_\epsilon^+;\ |t|\geqq\eta}\varphi(x,t)\sigma^{-2}dxdt\right],$$

puis les distributions analogues correspondant à V_ϵ^- et leurs combinaisons. On a par exemple $\bar{D}=(1/4\pi)\,\mathrm{Pl}\,s^{-2}$ (ce qui prouve le caractère invariant de $\mathrm{Pl}\,s^{-2}$ par le groupe de Lorentz, qui ne résulte pas immédiatement de la définition). Ensuite

$$\mathrm{P}\,\sigma^{-2}(\varphi)=\lim_{\epsilon\to 0}\left(\iiiint_{V_\epsilon^+}+\iiiint_{V_\epsilon^-}\right)\varphi\sigma^{-2}dxdt,$$

et $D^{(1)}=-(1/2\pi^2)\,\mathrm{P}\,\sigma^{-2}$. Enfin $-\Box\log|\sigma|=\mathrm{P}\,\sigma^{-2}$.

L. Schwartz (Paris).

Švec, M. E. On the solution of a problem for an equation of parabolic type. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 243–244 (1954). (Russian)

The equation associated with heat conduction or diffusion

$$\text{(1)}\qquad \frac{\partial}{\partial z}\left[z^\sigma\frac{\partial q}{\partial z}\right]+A_1(x)z^\sigma\frac{\partial q}{\partial z}=B_1(x)z^{\sigma-1}\frac{\partial q}{\partial x}\quad(\sigma<1)$$

is solved subject to the conditions

$$\text{(2)}\qquad q(0,z)=0,\quad q(x,0)=1,\quad q(x,\infty)=0.$$

After certain substitutions, equation (1) is reduced to the ordinary differential equation

$$\text{(3)}\qquad \frac{d^2f}{d\eta^2}+\frac{\eta^\beta}{1+\beta}\frac{df}{d\eta}=0,$$

whose solution satisfying the conditions $f(0)=1$ and $f(\infty)=0$ may be expressed in terms of the incomplete gamma function. For the case in which the coefficients A_1 and B_1 are related by a type of continuity equation, equation (1) is solved for a concentrated or point source.

C. G. Maple (Ames, Iowa).

Germain, Paul, et Bader, Roger. Sur le problème de Tricomi. Rend. Circ. Mat. Palermo (2) 2, 53–70 (1953).

Die Verfasser untersuchen zunächst eine dreiparametrige Gruppe der Differentialgleichung von Tricomi: $zu_{xx}+u_{zz}=0$, mittels welcher man weitere Lösungen dieser Differentialgleichung gewinnen kann, wenn eine spezielle solche vorliegt. Unter den Transformationen dieser Gruppe sind diejenigen T^+ von Bedeutung, die reell in der Halbebene $z>0$ als Produkte von Inversionen in der Halbebene x, $y>0$ aufgefasst werden können. Ihre Pole liegen auf der x-Achse. Damit ergeben sich Beziehungen zur Geometrie in Poincarés Halbebene. Auch Bedingungen, unter welchen die Transformationen T^+ in das Gebiet der reellen Halbebene $z<0$ fortgesetzt werden können, lassen sich angeben. Dabei bleibt das System der Charakteristiken erhalten und diese bilden die Grenze zwischen dem Gebiet der Transformationsklasse T^+ und einer zweiten solchen T^-. In der uneigentlich abgeschlossenen elliptischen Halbebene können die beiden Klassen zusammengefasst werden.

Wird Tricomis Differentialgleichung in die Form

$$3[r^2u_{rr}+(1-t^2)u_{tt}]+4(ru_r-tu_t)=0,\quad r^2=x^2+y^2,\quad rt=x$$

gebracht, so ergeben sich Lösungen, welche von solchen der Gaussschen hypergeometrischen Differentialgleichung aufgebaut werden können. In der Halbebene $z<0$ gehen sie (durch die Kummersche Transformation) in die Darbouxsche Lösung über. Die Verfasser bestimmen Gültigkeitsbereiche dieser Lösungen und erwähnen die Möglichkeiten einer Fortsetzung [cf. G. Guderley, Air Materiel Command, Wright Field, Tech. Rep. No. **F-TR-2168-ND** (1947)]. Die Anwendung der Gruppe erlaubt nunmehr von Darboux's Lösung neue Lösungen zu gewinnen, wie die Verfasser zeigen. Besondere zusätzliche Invarianzeigenschaften stellen sich ein, wenn der die hypergeometrische Funktion charakterisierende Parameter verschwindet. Die Riemannsche Funktion $R(M,P)$ des Problems ist dann in

jedem Punkt der (elliptischen) Halbebene bestimmt. Insbesondere ergeben sich Lösungen, die abgesehen vom Punkt M in der ganzen (x, z)-Ebene regulär verlaufen. Weiterhin konstruieren die Verfasser in eindeutiger Weise die Greensche Funktion für das Problem von Tricomi speziell auch im Falle, dass der Punkt M in der hyperbolischen Halbebene liegt. Zum Schluss wird das Problem von Tricomi für sogenannte Normalbogen entwickelt. Lösungen in diesen Fällen gestatten Verallgemeinerungen für beliebige einfache Bogenstücke. *M. Pinl* (Köln).

Huber, Alfred. A theorem of Phragmén-Lindelöf type. Proc. Amer. Math. Soc. 4, 852–857 (1953).

For the equation $L_k[u]=\sum_1^n\partial^2u/\partial x_i^2+kx_n^{-1}\partial u/\partial x_n=0$, $k<1$, the author proves the following theorem of Phragmén-Lindelöf type: Let u be a solution of $L_k[u]=0$, $k<1$, defined in the half-space $H(x_n>0)$ and satisfying at the boundary $D(x_n=0)$ the condition $\lim\sup_{P\to M}u(P)\leqq 0$ $(P\in H;\, M\in D)$. Then (a) the limit $\alpha=\lim_{r\to\infty}m(r)/r^{1-k}$ always exists (finite or infinite), where $m(r)=\sup u(P)$ on the hemisphere $\sum_1^n x_i^2=r^2$; (b) $\alpha\geqq 0$; (c) $u\leqq\alpha x_n^{1-k}$ throughout H, and if the equality is attained at any point in H, it holds identically. This theorem generalizes to the equation $L_k[u]=0$, the analogous result for harmonic functions due to M. Heins [Trans. Amer. Math. Soc. 60, 238–244 (1946); these Rev. 8, 371], and can be expected to play the same part in generalized axially symmetric potential theory as the classical Phragmén-Lindelöf theorem does in potential theory. Simple counterexamples show that the theorem cannot hold for $k\geqq 1$. *D. Gilbarg* (Stanford, Calif.).

Mysovskih, I. P. On a boundary problem for the equation $\Delta u=k(x, y)u^2$. Doklady Akad. Nauk SSSR (N.S.) 94, 995–998 (1954). (Russian)

Let u be a function which satisfies the differential equation (1) $\Delta u=k(x, y)u^2$ in a bounded, simply connected domain D and such that (2) $u|_S=f(s)$ on the boundary S of D. Here $k(x, y)>0$ and continuously differentiable in the closed domain $\bar{D}$ and $f(s)\geqq 0$, continuous, and $f(s)\not\equiv 0$ on S. The author introduces a function $\bar{u}$, harmonic in D and such that $\bar{u}|_S=f(s)$, for the purpose of changing the dependent variable through the substitution $v=u-\bar{u}$. This gives the boundary-value problem (3) $\Delta v=k(x, y)(v+\bar{u})^2$, $v|_S=0$, which is equivalent to the integral equation

$$(4)\quad v(x, y)+\iint_D G(x, y: \xi, \eta)k(\xi, \eta)\times[v(\xi, \eta)+u(\xi, \eta)]^2d\xi d\eta=0,$$

where G is a Green's function. This integral equation is solved by Newton's method starting with the initial approximation $v_0\equiv 0$. It is proved that the approximating sequence so defined converges to the solution of (4), which leads to a solution of the original boundary-value problem. *C. G. Maple* (Ames, Iowa).

Szegö, G. Inequalities for certain eigenvalues of a membrane of given area. J. Rational Mech. Anal. 3, 343–356 (1954).

In the discussion of the normal modes of vibration of a membrane about its position of equilibrium, it is necessary to find the values of a constant $h(\geqq 0)$, the eigenvalues, such that the equation

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+h^2u=0$$

has a non-zero solution in a domain D bounded by a simple analytic curve C on which either (i) $u=0$ or (ii) $\partial u/\partial n=0$. In both cases infinitely many eigenvalues exist; in case (i) they are denoted by $\lambda_1\leqq\lambda_2\leqq\lambda_3\leqq\cdots$, in case (ii) by $\mu_1\leqq\mu_2\leqq\mu_3\leqq\cdots$. It is well-known that $0<\lambda_1<\lambda_2$ and that $\mu_1=0$; and also that for a circle of radius a, $\lambda_1=j/a$, $\mu_2=\mu_3=p/a$ where $j=2.4048\cdots$ is the smallest positive zero of $J_0(r)$ and $p=1.8412\cdots$ the smallest positive zero of $J_1'(r)$.

It was conjectured by Lord Rayleigh [Theory of sound, vol. I, 2nd ed., Macmillan, London, 1894, p. 339] that, for a domain of area A, $\lambda_1\geqq j(\pi/A)^{1/2}$, with equality for a circle, a result proved by Faber [S.-B. Math.-Phys. Kl. Bayer. Akad. Wiss. 1923, 169–172] and by Krahn [Math. Ann. 95, 97–100 (1925)]. It has recently been conjectured by Kornhauser and Stakgold [J. Math. Physics 31, 45–54 (1952); these Rev. 13, 846] that $\mu_2\leqq p(\pi/A)^{1/2}$, again with equality for a circle. The first purpose of this paper is to prove this conjecture. The proof is based on the minimum property

$$\mu_2^2=\min\frac{\iint_D|\text{grad }u|^2d\sigma}{\iint_D u^2d\sigma}$$

admitting in this problem all functions u defined in D such that $\iint_D u\,d\sigma=0$, where $d\sigma$ is the area-element of D. The method is to "transplant" solutions of the problem for the unit circle to the general domain by means of a schlicht conformal mapping of D on the unit circle.

The last part of the paper deals with "nearly circular" domains D, for which second variation of the eigenvalues λ_1, λ_2, λ_3, λ_2, λ_3 is computed and compared with the area and other functionals of D. The proofs in the second part are omitted; for them, reference is made to G. Szegö's Tech. Rep. 34, Office of Naval Research, Contract N6ori-106, Task order 5 (NR-43-992), Stanford Univ., Calif. *E. T. Copson* (St. Andrews).

Minakshisundaram, S. Eigenfunctions on Riemannian manifolds. J. Indian Math. Soc. (N.S.) 17 (1953), 159–165 (1954).

Let Δ be the Laplace operator on a connected, compact, orientable and C^∞ Riemannian manifold, and let $\{\lambda_n\}$ and $\{\varphi_n(x)\}$ be the eigenvalues and the corresponding eigenfunctions of Δ. Jointly with A. Pleijel [Canadian J. Math. 1, 242–256 (1949); these Rev. 11, 108], the author discussed the behaviour of the Dirichlet series

$$\text{(i)}\quad \sum_n\lambda_n^{-s}\varphi_n(x)\varphi_n(y)\quad\text{and}\quad\text{(ii)}\quad\sum_n\lambda_n^{-s}\varphi_n(x)^2,$$

by generalizing Carleman's method [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. 88, 119–132 (1936)]. The result was applied, by the help of Ikehara's Tauberian theorem, to deduce asymptotic distributions of λ_n and of $\varphi_n(x)$. In the present paper, the author devises an alternative and simplified proof of the behaviours of (i) and (ii). It does not appeal to Carleman's analysis. Instead, the author makes use of the asymptotic behaviour at $t=+0$ of the fundamental solution

$$\text{(iii)}\qquad U(x, y, t)=\sum_n\varphi_n(x)\varphi_n(y)\exp(-\lambda_nt)$$

of the heat equation $\partial/\partial t=\Delta$. Starting with the parametrix for the heat equation, obtained previously in the above joint paper, the fundamental solution (iii) is constructed by a method of iteration [cf. the reviewer, Osaka Math. J. 5,

65–74 (1953); these Rev. 15, 36]. This procedure itself enables the author to obtain the above mentioned asymptotic behaviour of (iii) at $t=+0$. *K. Yosida.*

✶Aronszajn, N. Green's functions and reproducing kernels. Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 355–411. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

Systems $\{A;\Lambda_i\}$ are considered where A is an elliptic linear differential operator with analytic coefficients, regular in a simply connected closed plane domain $\bar{D}$ in the (x, y)-plane with analytic boundary, Λ_i are linear differential boundary operators with analytic coefficients regular in the arc-length parameter on the boundary C. Such a system is called a formally positive system if for all $u \in C^{(m)}$ ($m=$order of A) and $\Lambda_i u=0$ on C there exists a representation:

$$(1)\quad \iint_D Au\bar{u}dxdy=\sum_{k=1}^{p}\iint_D |A_k u|^2dxdy+\sum_{j=1}^{q}\int_C|\Omega_j u|^2ds,$$

where A_k and Ω_j are operators in D and on C, respectively. In this case $m=2t$ is even, and there exists a decomposition (2) $A=\sum A_k{}^*A_k$ where $A_k{}^*$ is the "adjoint" of A_k. (1) and (2) are not uniquely determined. One demands also that the order of the Ω_j is $\leqq t-1$.

A non-empty finite system of boundary operators $\{\Lambda_j'\}$, is called weaker than $\{\Lambda_i\}$ if for every proper sub-arc $C'\subset C$ and for every function $u \in C^{(m')}$ (m' large enough) $\Lambda_i u=0$ on C' implies $\Lambda_j'u=0$ on C'. The two systems are equivalent if each one is weaker than the other. A system $\{\Lambda_i\}$ is called normal if the normal order of Λ_i is strictly increasing and

$$\Lambda_i=\frac{\partial^{l_i}}{\partial n^{l_i}}+\sum_{k=0}^{l_i-1}\theta_{ik}\frac{\partial^{k_i}}{\partial n^{k_i}},$$

where θ_{ik} are pure tangential operators with order $\leqq l_i-k_i$. Conditions in order that $\{\Lambda_j'\}$ be weaker than (equivalent to) $\{\Lambda_i\}$ and basic properties of normal systems are described. The formula

$$(3)\quad \iint_D Au\bar{v}dxdy=\sum_k\iint_D A_kuA_k\bar{v}dxdy$$
$$+\sum\int_C\Omega_ju\overline{\Omega_jv}ds+\int_C\Gamma(u, v)ds,$$

where Γ is a bilinear operator of total order $2t-1$, holds for $uv \in C^{(2t)}$. In fact, (3) holds for a somewhat larger class. If $\{\phi_k\}$ is a normal system, $k=0, 1, \cdots, t-1$, and ϕ_k is of order k, then there exists a normal system $\{\psi_{2t-1-k}\}$, $k=0, 1, \cdots, t-1$, (the index denotes the order) such that for all $u \in C^{(2t-1)}$, $v \in C^{(t-1)}$,

$$\int_C\Gamma(u, v)ds=\int_C p(s)\sum_{k=0}^{t-1}(-1)^k\psi_{2t-1-k}u\overline{\phi_kv}\,ds,$$

$$p(s)=\sum_{i=0}^{m}a_i(s)[\alpha(s)]^i[\beta(s)]^{m-i},$$

where $\alpha(s)$, $\beta(s)$ are the direction cosines of the exterior normal to C and $A^0=\sum_{i=0}^m a_i(s)\partial^m/\partial x^i\partial y^{m-i}$ is the leading part of A. Let k' be any subsequence of $\{k\}=\{0, 1, \cdots, t-1\}$ and k'' its complement; then the system formed by $\phi_{k'}$ and $\psi_{2t-1-k''}$ is called a minimal system with respect to A_k and Ω_j. If $\{A;\Lambda_i\}$ is formally positive and $\{\Lambda_i\}$ is normal, then there exists a minimal system $\{\Lambda_j'\}$ weaker than $\{\Lambda_i\}$; every such minimal system forms with A a formally positive system and no normal system of boundary operators strictly weaker than a minimal system can form with A a formally positive system. Other properties in this connection are given, followed by a detailed analysis of the cases: $A=-\Delta$, $-\Delta-\mu I$ (μ real), Δ^2.

A Green function relative to the formally positive-definite system $\{A;\Lambda_i\}$ is introduced and under the assumption that it exists and also satisfies certain regularity conditions, basic properties are deduced. The notion of reproducing kernel, i.e., the Bergman kernel function for Hilbert spaces [The kernel function and conformal mapping, Math. Surveys, No. 5, Amer. Math. Soc., New York, 1950; these Rev. 12, 402] and its properties are described [see Aronszajn, Trans. Amer. Math. Soc. 68, 337–404 (1950); these Rev. 14, 479]. It is shown that for $t>1$ the Green function is a reproducing kernel for a certain class of functions $u \in \bar{K}\supset K$ ($u \in K$ if $u \in C^{(2t)}$ in $\bar{D}$ and $\Lambda_iu=0$) with the norm (1). This class does not form a complete space; hence, by the author's method of functional completion (which is described and illustrated by examples) the "best" completion of $\bar{K}$ (or K) is found to exist and to be the space $\tilde{K}$ of all functions u which are limits everywhere of Cauchy sequences in K. The space $\tilde{K}$ has many properties which may characterize it. By using them one obtains a generalization of the Bergman-Schiffer formula [Duke Math. J. 15, 535–566 (1948); these Rev. 10, 42]:

$$K(z, \zeta)=\frac{1}{2\pi}[N(z, \zeta)-G(z, \zeta)],$$

where K, N, G are the kernel function, the Neumann function, and the Green function in the case of elliptic positive operators of the second order. This yields applications to the theory of boundary-value and eigenvalue problems for $\{A;\Lambda_i\}$.

In certain instances it is possible to prove the existence of Green functions by verifying the existence and some properties of $\tilde{K}$. This is illustrated by the example $\{\Delta^2+I;\Delta, \partial\Delta/\partial u\}$; however in this case the regularity conditions for Green functions are not satisfied. There are indications as to how one can generalize the theory to the case of several variables and how it is possible to weaken the assumptions on D and on A and Λ_i. *M. Maschler.*

Functional Analysis

✶Zaanen, Adriaan Cornelis. Linear analysis. Measure and integral, Banach and Hilbert space, linear integral equations. Interscience Publishers Inc., New York; North-Holland Publishing Co., Amsterdam; P. Noordhoff N.V., Groningen, 1953. vii+601 pp. $11.00.

Part I (87 pages) deals with measure and integral, the Lebesgue function spaces L_p and the Orlicz spaces L_Φ. It gives a concise discussion of these subjects, complete in itself but leaving aside everything which seemed unessential for the later parts of the book. As stated in the preface, free use was made in this part of lecture notes of N. G. de Bruijn. The integral is defined as a product measure in the product space of the underlying measure space and the straight line.

Part II (279 pages) is devoted to the general theory of linear transformations in Banach space and Hilbert space. It begins with the axiomatic definition of these spaces and their most obvious properties, including a discussion of product spaces and factor spaces, and such examples as the spaces L_p, L_Φ, l_p, and the space of B_p-almost periodic func-

tions. There follows a chapter on linear functionals and linear transformations in Banach spaces, weak convergence, adjoint space, adjoint transformations, projections, including a neat exposition of Banach's theorems on closed linear transformations. [The adjoint space E^* of a real or complex Banach space E is defined as the space of all bounded linear functionals on E. It appears to the reviewer that, in the complex case, it is more convenient to define E^* as the space of all bounded conjugate-linear functionals on E since, in the particular case of a complex Hilbert space E, E^* can then be identified with E, and so it is then superfluous to distinguish, as the author does, between the "Banach-adjoint" and the "Hilbert-adjoint" of a transformation in Hilbert space.]

The linear transformations of finite-dimensional spaces are treated in a separate chapter, yielding a concise exposition of the canonical form of a matrix, and of elementary divisors. There follows a chapter on bounded linear transformations in Hilbert space, which treats in particular the integral transformations, especially those with kernels of finite double-norm, and expounds the Fredholm theory for these transformations in the form given to it by F. Smithies [Duke Math. J. **8**, 107–130 (1941); these Rev. **3**, 47]. Symmetric, unitary, normal and symmetrizable transformations are introduced, but the spectral theory of bounded normal transformations is not included.

There follows a chapter on the mutual relations between the ranges and null spaces of a bounded linear transformation and its adjoint, and on the spectrum and the resolvent. Compact (=vollstetig) linear transformations are introduced in the following chapter which expounds in particular the theory of F. Riesz, facts about the resolvent, and ergodic theorems of K. Yosida [Proc. Imp. Acad. Tokyo **14**, 292–294 (1938)]. The compact linear transformations of Hilbert space, which are symmetric, or normal, or symmetrizable, are treated very thoroughly in the following chapter. The theory of compact symmetrizable linear transformations, which originates in papers of J. Marty [C. R. Acad. Sci. Paris **150**, 1499–1502 (1910)], A. J. Pell [Trans. Amer. Math. Soc. **12**, 165–180 (1911)], and A. Garbe [Math. Ann. **76**, 527–547 (1915)] on integral transformations with symmetrizable kernel, was developed very considerably in the last 12 years, mainly by contributions of the author, W. T. Reid, and H. J. Zimmerberg, so that we possess now fairly complete analogues, for these transformations, of the expansion theorems and minimax theorems of compact symmetric transformations.

Part III deals with non-singular linear integral equations, i.e. with the Fredholm theory and spectral theory of linear integral transformations one of whose iterates is compact. The first chapter generalizes the Fredholm-Smithies theory to integral equations in arbitrary spaces L_p and L_ϕ. The second one deals with expansion theorems for integral equations with normal, Hermitian, or positive-definite kernel. The last two chapters are devoted to a thorough study of integral equations with a symmetrizable kernel.

A rich collection of examples and particular theorems illustrates and completes the abstract developments. Compared to the richness of the material dealt with, the bibliographical references are rather sparse. Though each chapter is concluded by a list of some references, these are far from complete, and one misses particularly the references for most of the theorems given as "examples".

The exposition is always clear and readable. The book is a valuable contribution to the growing literature of textbooks on Functional Analysis. Its main feature is the emphasis laid on integral equations and especially on those with symmetrizable kernel, a domain of research in which we owe to the author many personal results.

B. Sz.-Nagy (Szeged).

da Silva Dias, C. L. Topological vector spaces and their application in analytic functional spaces. Bol. Soc. Mat. São Paulo **5** (1950), 1–58 (1952). (Portuguese)

This is the author's thesis [São Paulo, 1951] and was reviewed in these Rev. **13**, 249.

Grothendieck, A. Résumé des résultats essentiels dans la théorie des produits tensoriels topologiques et des espaces nucléaires. Ann. Inst. Fourier Grenoble **4** (1952), 73–112 (1954).

This paper summarizes the author's results on tensor products of locally convex topological linear spaces, which will appear with full details and proofs in Mem. Amer. Math. Soc. Tensor products of Banach spaces had previously been studied by Schatten and von Neumann [cf. R. Schatten, A theory of cross spaces, Princeton, 1950; these Rev. **12**, 186], but the author's work goes far beyond these earlier studies, even for Banach spaces. No adequate summary of such a summary is possible here; only some highlights will be touched on.

For locally convex spaces E and F, several topologies on their algebraic tensor product $E\otimes F$ are considered; in case E and F are normed spaces, $E\otimes F$ is given norms, agreeing with these topologies, which were already studied by Schatten. The completions of $E\otimes F$ under these various topologies, or even norms, are the (topological!) tensor products considered by the author; the two most important of these are denoted, respectively, by $E\hat{\otimes}F$ and $E\check{\otimes}F$. The topology on $E\otimes F$ induced by $E\hat{\otimes}F$ is finer than that induced by $E\check{\otimes}F$, and hence there exists a natural continuous linear transformation $u: E\hat{\otimes}F\rightarrow E\check{\otimes}F$. The most important unsolved problem of the paper is whether u is always one-to-one; this problem is shown to be equivalent to many others, including Banach's problem of approximating completely continuous linear transformations between Banach spaces by ones with finite-dimensional range. One of the many theorems proved about these tensor products is that, if L^1 and L_E^1 denote, respectively, the spaces of integrable functions on a measure space with real values and with values in a Banach space E, then there is a natural linear isometry between L_E^1 and $L^1\hat{\otimes}E$.

About half the paper is devoted to locally convex spaces E with the property that, for every locally convex space F, $E\hat{\otimes}F=E\check{\otimes}F$ (algebraically and topologically); such spaces are called nuclear. The space of analytic functions on an analytic manifold (compact—open topology), and many of the spaces studied in L. Schwartz's theory of distributions, are nuclear. (Schwartz's "théorème des noyaux" essentially asserts the nuclearity of his spaces $\mathcal{E}(R^m)$.) A Banach space is nuclear if and only if it is finite-dimensional; an $\mathfrak{F}$-space is nuclear if and only if every unconditionally convergent series in it converges absolutely. (This yields a new proof of the Dvoretzky-Rogers theorem [Proc. Nat. Acad. Sci. U. S. A. **36**, 192–197 (1950); these Rev. **11**, 525].) Every bounded subset of a nuclear space is totally bounded. Every nuclear space is isomorphic to a subspace of a cartesian product of Hilbert spaces. Finally, every subspace, quotient space, or cartesian product of nuclear spaces is nuclear. [Reviewer's note: The first result in footnote 4

remains true with no assumptions on M whatever. The following misprints were noted: p. 82, lines 12 and 16, for $\mathfrak{L}$ read $\mathfrak{B}$; p. 100, line 29, for F' read F_r'; p. 101, line 1, for Banach read Banach F.] *E. Michael.*

Alexiewicz, A. On the two-norm convergence. Studia Math. 14 (1953), 49–56 (1954).

Following Fichtenholz [Mat. Sbornik N.S. 4(46), 193–214 (1938)] and earlier notes of his own [Studia Math. 11, 1–30 (1949); 200–236 (1950); these Rev. 12, 418, 507], the writer discusses two-norm, or γ, convergence: Given a space with two norms $\|\cdot\cdot\|$ and $\|\cdot\cdot\|^*$ such that $\|x_n\|\to 0$ implies $\|x_n\|^*\to 0$, say that $x_n\to_\gamma x$ if $\|x_n\|$ is bounded and $\|x_n-x\|^*\to 0$. Examples are given of such spaces with or without various other properties. *M. M. Day* (Urbana, Ill.).

Halperin, Israel. Uniform convexity in function spaces. Duke Math. J. 21, 195–204 (1954).

Halperin, Israel. Reflexitivity in the L^λ function spaces. Duke Math. J. 21, 205–208 (1954).

Continuing the study of function spaces defined in two earlier papers [Halperin, Canadian J. Math. 5, 273–288 (1953); Ellis and Halperin, ibid. 5, 576–592 (1953); these Rev. 15, 38, 439] the author gives conditions on functions λ or w such that L^λ or $L_w{}^p$ be uniformly convex (uc) or reflexive. It is also shown that $L^\lambda(B)$ is reflexive (or uc) if and only if both L^λ and B are reflexive (uc). [The uc case was proved by the reviewer in Bull. Amer. Math. Soc. 49, 745–750 (1943); these Rev. 5, 146.] *M. M. Day.*

Novotný, Miroslav. Le noyau abstrait de la construction de Weyr des nombres caractéristiques des matrices. Publ. Fac. Sci. Univ. Masaryk 1953, 41–51 (1953). (Czech. Russian and French summaries)

In an abstract space, without algebraic operations but with a linear dependence relation axiomatized as in a paper by Haupt, Nöbeling and Pauc [J. Reine Angew. Math. 181, 193–217 (1940); these Rev. 1, 169], there is obtained a generalization of the Weyr characteristic numbers of a linear operator in a vector space. The space is required to have the property that dimensions of subspaces behave as in a projective space (dim T_1+dim T_2=dim $(T_1\cup T_2)$+dim $(T_1\cap T_2)$) and the operator F is required to be closed, continuous, and such that if T_0 and T are invariant subspaces with T properly containing T_0 then there is a point $x\in T-T_0$ such that $F(x)$ is in the subspace spanned by T_0 and x. [The review is based on the French summary.] *W. Givens.*

Phillips, R. S. A note on the abstract Cauchy problem. Proc. Nat. Acad. Sci. U. S. A. 40, 244–248 (1954).

The abstract Cauchy's problem due to E. Hille is discussed from the author's extension of the semi-group theory [Ann. of Math. (2) 59, 325–356 (1954); these Rev. 15, 718]. A semi-group $S(t)$ is said by the author to be of class (C_0) if strong $\lim_{t\downarrow 0} S(t)x=x$; it is of class $(0, A)$ if $\int_0^1\|S(t)x\|dt<\infty$ and if strong $\lim_{\lambda\to\infty} \lambda\int_0^\infty \exp(-\lambda t)S(t)xdt=x$. The author defines and discusses the ACP$_1$ (or ACP$_2$) for a linear operator U with domain and range dense in a Banach space. He thus defines the problem of finding a function $y(t)=y(t; y_0)$ satisfying the conditions: (i) $y(t)$ is strongly absolutely continuous and continuously differentiable in each finite subinterval of $[0, \infty)$ (or $(0, \infty)$), (ii) $y(t)\in D(U)$ and $U(y(t))=y'(t)$ for $t>0$, (iii) strong $\lim_{t\downarrow 0} y(t)=y_0$. ACP$_1$ is very similar to the abstract Cauchy problem discussed by E. Hille [Ann. Soc. Polon. Math. 25, 56–68 (1953); Ann. Inst. Fourier Grenoble 4, 31–48 (1954); these Rev. 15, 39, 718]. The Theorem 3.2 states the following: Let $\|R(\lambda; U)\|=O(\lambda^{-1})$ as $\lambda\to\infty$ for the resolvent of a closed linear operator U with dense domain, and for each $y_0\in D(U)$ let there be a unique solution to ACP. Then U generates a semi-group $S(t)$ of class $(0, A)$ such that $S(t)y=y(t; y_0)$ for all $y_0\in D(U)$. *K. Yosida* (Princeton, N. J.).

Hille, Einar. Quelques remarques sur les équations de Kolmogoroff. Bull. Soc. Math. Phys. Serbie 5, no. 3-4, 3–14 (1953).

Let $A=(a_{ij})$ denote an infinite-dimensional matrix satisfying

$$a_{jk}\geqq 0\ (j\neq k),\quad a_{jj}\leqq 0,\quad \sum_k a_{jk}=0.$$

The author integrates the pair of Kolmogoroff's equations

$$(1)\qquad Y'(t)=AY(t),\quad Z'(t)=Z(t)A,$$

with the help of the semi-group theory. In an earlier paper [Proc. Nat. Acad. Sci. U. S. A. 40, 20–25 (1954); these Rev. 15, 706], the author integrated (1) for the special case of triangular matrices $A=(a_{jk})$ ($a_{jk}=0$ if $k>j$) in the Markoff algebra of matrices $B=(b_{jk})$, metrized by $\|B\|=\sup_j\sum_k|b_{jk}|<\infty$. In the present paper, the author discusses (1) from the view-point of the perturbation theory of R. S. Phillips [Trans. Amer. Math. Soc. 74, 199–221 (1953); these Rev. 14, 882] and the null solution $Y(t)=0$ satisfying $\lim_{t\downarrow 0} y_{jk}(t)=0$, discovered by the author (the paper cited above). The discussions are related also to the case of the Banach space (l). *K. Yosida.*

***Schmeidler, Werner. Lineare Operatoren im Hilbertschen Raum.** B. G. Teubner, Stuttgart, 1954. vi+89 pp. DM 7.80.

The book is divided into three chapters. The first chapter presents a detailed discussion of sequential Hilbert space, and, motivated thereby, gives a general definition of Hilbert space. One of the conditions of the general definition is (in effect) that the dimension of the space be exactly $\aleph_0$. The concept of weak (sequential) convergence receives some emphasis.

The second chapter treats operators; most of it is devoted to completely continuous operators. Since most of the results are special cases of the Riesz theory of such operators, the student with an inclination to modern abstraction might find Banach's book an easier source for this material. [Incidentally, in the preface the author mentions the following theorem as possibly new: a completely continuous operator A has no non-zero proper values if and only if $\lim_n \|A^nx\|^{1/n}=0$ for all x. The result follows, in the Riesz theory, from the fact that a non-zero complex number is in the spectrum of a completely continuous operator if and only if it is a proper value, and from the admittedly nontrivial fact that the spectral radius of an operator A is equal to $\lim_n \|A^n\|^{1/n}$. It might also be mentioned that the author's statements about spectra look different from such statements in most other modern treatments of the subject; the difference is caused by defining the spectrum of A in terms of $I-\lambda A$ instead of $A-\lambda I$.]

The third chapter treats the spectral theorem. For a bounded Hermitian operator A and a real number λ, the author defines $E(\lambda)$ as the projection on the null space of the positive part of $A-\lambda I$. The construction of positive parts is based on the existence of square roots, proved in the second chapter by means of the binomial series. The spectral theorem is then extended to bounded normal operators and to unbounded self-adjoint operators by familiar reductions

to the bounded Hermitian case. The treatment is rather condensed.

The book ends with a brief section on examples and applications (e.g., to crystallography and quantum mechanics).

P. R. Halmos (Chicago, Ill.).

Harazov, D. F. On a class of linear equations with symmetrizable operators. Doklady Akad. Nauk SSSR (N.S.) 91, 1023–1026 (1953). (Russian)

The author considers equations of the form

$$(1)\qquad (E-\lambda A_1-\lambda^2 A_2)x=y,$$

where A_1 and A_2 are linear operators of finite norm in a complex Hilbert space and E is the identical operator; A_1 and A_2 are assumed symmetrizable, in the sense that there exists a positive definite operator H of finite norm such that $P_1=HA_1$ and $P_2=HA_2$ are Hermitian. He shows that, if A_2 has only positive characteristic values, then all the characteristic values of the associated homogeneous equation

$$(2)\qquad (E-\lambda A_1-\lambda^2 A_2)x=0$$

of (1) are real; and that if x_1 and x_2 are characteristic vectors of (2) belonging to distinct characteristic values λ_1 and λ_2, then they satisfy the orthogonality relation

$$(Hx_1, x_2)+\lambda_1\lambda_2(P_2x_1, x_2)=0.$$

A full orthogonal system of characteristic vectors is constructed, and expansion theorems for P_1f and P_2f, where f is an arbitrary vector, are given in terms of weak convergence. If λ is not a characteristic value of (2), the solution x of (1) can be obtained from the expansion

$$H(x-y)=\lambda\sum_{\nu=1}^{\infty}\frac{(y,Hx_\nu)}{\lambda_\nu-\lambda}Hx_\nu,$$

which is weakly convergent. Conditions are obtained for these expansions to be strongly convergent in various norms.

F. Smithies (Cambridge, England).

Harazov, D. F. On the theory of symmetrizable operators with polynomial dependence upon a parameter. Doklady Akad. Nauk SSSR (N.S.) 91, 1285–1287 (1953). (Russian)

The author considers equations of the form

$$(1)\qquad x-\sum_{\nu=0}^{m}\lambda^\nu A_\nu x=0,$$

where the A_ν are compact linear operators in a complex Hilbert space, symmetrizable in the sense that there is a compact positive definite linear operator H such that all the operators $P=HA$ are self-adjoint. The following results are proved. If A_0 has no positive characteristic values $\leqq 1$ (and, if m is even, A_m has at least one positive characteristic value), then there exists at least one real characteristic value of (1). If unity is not a characteristic value of A_0, the characteristic values of (1) have no finite limit point. If A_0 has no positive characteristic values $\leqq 1$, and $A_2, \cdots, A_m$ have only positive characteristic values, then (1) has no characteristic values λ such that

$$0<\arg\lambda\leqq\frac{\pi}{m-1}\quad\text{or}\quad 2\pi-\frac{\pi}{m-1}\leqq\arg\lambda<2\pi;$$

these ranges are best possible. Additional results are given for the special case $m=2$.

F. Smithies.

Leżański, T. Sur les fonctionnelles multiplicatives. Studia Math. 14 (1953), 13–23 (1954).

If $\mathfrak{A}$ is a linear normed ring, $\Phi(A)$ is a linear function on $\mathfrak{A}$, and a set of functions $a_n(A)$ and operations $U_n(A)$ on $\mathfrak{A}$ to $\mathfrak{A}$ are defined sequentially by the conditions

$$a_0(A)=1,\quad U_1(A)=A,\quad U_n(A)=a_{n-1}(A)\cdot A-U_{n-1}(A)A,$$
$$a_n(A)=n^{-1}\Phi(U_n(A)),$$

then $D_\lambda(A)=\sum_{n=0}^{\infty}\lambda^n a_n(A)$ and $U_\lambda(A)=\sum_{n=0}^{\infty}\lambda^n U_n(A)$ are generalized Fredholm determinants [see T. Lalesco, Théorie des équations intégrales, Hermann, Paris, 1912, pp. 25, 26]. It is proved that if $A_1A_2=A_2A_1$ and $\|A_i\|<1/5$, $i=1, 2$, then $D(A_1+A_2+A_1A_2)=D(A_1)D(A_2)$. This result is applied to the Fredholm determinants defined in an earlier paper [Studia Math. 13, 244–276 (1953); these Rev. 15, 535].

T. H. Hildebrandt (Ann Arbor, Mich.).

Sikorski, R. On Leżański's determinants of linear equations in Banach spaces. Studia Math. 14 (1953), 24–48 (1954).

This is a detailed exposition of the author's note in Bull. Acad. Polon. Sci. Cl. III. 1, 219–221 (1953); these Rev. 15, 719. It extends the product theorem of the paper reviewed above to the case when linear function Φ on $\mathfrak{A}$ satisfies the condition $\Phi(A_1A_2\cdots A_n)=\Phi(A_2A_3\cdots A_nA_1)$ for any $A_1\cdots A_n$ of $\mathfrak{A}$. The author also summarizes the Leżański Fredholm determinant theory and gives a more satisfactory setting for the adjoint equation.

T. H. Hildebrandt.

Reiter, H. J. On a certain class of ideals in the L^1-algebra of a locally compact abelian group. Trans. Amer. Math. Soc. 75, 505–509 (1953).

This is an addition to an earlier paper of the author [same Trans. 73, 401–427 (1952); these Rev. 14, 465]. Let G be a locally compact Abelian group. It is proved, following a device of Bochner [Math. Ann. 96, 119–147 (1926)], that if I is a closed ideal in $L^1(G)$ of which the hull is countable and consists of independent elements, and if $\varphi \in L^\infty$ is orthogonal to I, then $\varphi=\sum_1^\infty a_n\alpha_n$, where $\{\alpha_n\}\subset$hull (I), and $\|\varphi\|_\infty=\sum|a_n|$. It then follows that the quotient algebra L^1/I is isometric to the algebra C (hull (I)) of all continuous complex-valued functions defined and vanishing at infinity on hull (I). [The "distance theorem" referred to is equivalent to the standard theorem relating the norm on a Banach space to that on its conjugate space.]

L. H. Loomis.

Nakamura, Masahiro, and Takeda, Zirô. Normal states of commutative operator algebras. Tôhoku Math. J. (2) 5, 109–121 (1953).

Suite d'articles antérieurs [même J. (2) 4, 275–283 (1952); Proc. Japan Acad. 28, 558–563 (1952); Kôdai Math. Sem. Rep. 1953, 23–26; ces Rev. 14, 1096, 991; 15, 204]. Les auteurs caractérisent, à une équivalence unitaire près, une C^*-algèbre par son spectre, supposé simple, et par une ou plusiers mesures sur ce spectre. Résultats analogues pour les opérateurs hermitiens.

J. Dixmier (Paris).

Bonsall, F. F., and Goldie, A. W. Annihilator algebras. Proc. London Math. Soc. (3) 4, 154–167 (1954).

The authors study a new class of Banach algebras, called annihilator algebras, which generalize the dual Banach algebras of Kaplansky [Ann. of Math. (2) 49, 689–701 (1948); these Rev. 10, 7]. An annihilator algebra is a complex Banach algebra such that the right and left annihilators of the whole algebra are both (0), while the right (resp. left) annihilator of every proper closed left (resp. right) ideal

is (0). Following Kaplansky, the authors obtain a satisfactory structure theory for their algebras, and obtain some results about them. Unfortunately, it is as yet unknown whether every annihilator algebra automatically satisfies the stronger requirements of a dual algebra.

E. Michael (Seattle, Wash.).

Amemiya, Ichiro. A generalization of Riesz-Fischer's theorem. J. Math. Soc. Japan **5**, 353–354 (1953).

H. Nakano has shown that a partially ordered normed linear space is complete if $x_i \geqq 0$ and $\sum \|x_i\| < \infty$ together imply (i) there exists $x = \sup(x_1 + \cdots + x_n)$ and (ii) $\|x\| \leqq \sum \|x_i\|$. The present author shows that (i) alone is sufficient. For: Nakano's proof continues to hold if (ii) is replaced by the weaker (ii)′ $\|x\| \leqq n \sum \|x_i\|$ for some n independent of the sequence x_i; and if (ii)′ were false for every n then there would be $y_{n,m} \geqq 0$ with $\sum_m \|y_{n,m}\| < 2^{-n}$ and $\|\sum y_{n,m}\| > n$. Then the resulting sequence $x_n = y_{n,1} + \cdots + y_{n,n}$ would contradict (i).

I. Halperin (Utrecht).

$\|\sum_m y_{n,m}\| > n$.

"$x_n = \sum_{i=1}^{n} y_{i,n+1-i} = y_{n,1} + y_{n-1,2} + \cdots + y_{1,n}$".

Tomita, Minoru. On the regularly convex hull of a set in a conjugate Banach space. Math. J. Okayama Univ. **3**, 143–145 (1954).

This note gives the special cases, in which X is the conjugate of a Banach space in its w^*-topology, of the following theorems, and applies theorem 2 to the special case $X = A^*$, A a C^*-algebra, K = set of states of A of norm one.

Theorem 1. Let X be a locally convex linear topological space and let N be a compact subset of X; then p belongs to the closed convex hull of N if and only if there is a normalized non-negative Borel measure φ of total weight 1 on N such that p is the weak integral $p = \int_N \lambda d\varphi(\lambda)$. Theorem 2. If X is as above and K a convex compact subset of X and N = closure of set of extreme points of K, then there is a φ as above such that $p = \int_N \lambda d\varphi(\lambda)$.

M. M. Day.

Vulih, B. Z. On imbedding a normed partially ordered space in its second conjugate space. Uspehi Matem. Nauk (N.S.) **9**, no. 1(59), 91–99 (1954). (Russian)

Let X be a (real) vector lattice which is normed and in which $|x| < |y|$ implies $\|x\| \leqq \|y\|$. A type of order convergence is introduced in X. Let $\{x_\alpha\}_{\alpha \in A}$ be a well-ordered transfinite sequence of elements of X (A is any well-ordered set). Then x_α converges transfinitely to $x \in X$ if there exist well-ordered sequences $\{y_\alpha\}_{\alpha \in A}$ and $\{z_\alpha\}_{\alpha \in A}$ which are $\uparrow$ and $\downarrow$ respectively, with $\inf_{\alpha \in A} z_\alpha = \sup_{\alpha \in A} y_\alpha = 0$, and such that $z_\alpha \leqq x_\alpha - x \leqq y_\alpha$ for all $\alpha \in A$. A linear functional on X is said to be completely linear if it is continuous in the sense of transfinite convergence. Let X^* be the space of all norm-continuous linear functionals on X; X^{**} similarly. The following theorems are proved. I. Every linear functional F_x on X^* of the form $F_x(x^*) = x^*(x)$ (for $x \in X$) is completely linear. II. Every functional $x^* \in X^*$ is completely linear if and only if $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \to 0$, for all well-ordered transfinite sequences $\{x_\alpha\}$. III. X is reflexive if and only if X and X^* have the property that $x_n \downarrow 0$ implies $\|x_n\| \to 0$ and $x_n \uparrow \infty$ implies $\|x_n\| \to \infty$ $(n = 1, 2, 3, \cdots)$. A number of other results, dealing with the properties of X considered as a subspace of X^{**} are also obtained. Most of the results of this paper have been previously obtained (some in a weaker form) by Nakamura [Tôhoku Math. J. (2) **1**, 100–108 (1949); these Rev. **11**, 186] and by Ogasaware [J. Sci. Hiroshima Univ. Ser. A. **12**, 37–100 (1942); **13**, 41–161 (1944); these Rev. **10**, 545].

E. Hewitt.

Theory of Probability

von Schelling, Hermann. Coupon collecting for unequal probabilities. Amer. Math. Monthly **61**, 306–311 (1954).

The problem considered is as follows: events E_i, $i = 1$ to k, occur with probabilities p_i; what is the probability that exactly r different kinds of events are found for the first time on the nth trial? This paper is said to be an expanded version of an earlier paper [Deutsch. Statist. Zentralbl. **26**, 137–146 (1934)] and gives the probability above in a symmetric function notation and an inclusion-exclusion form (a binomial coefficient $\binom{k-1}{m-1}$ is missing in its last term) as well as its first and second factorial moments. It is noticed that its cumulation, for successive values of n, has a simpler formula; but this is just the probability of at least r kinds in n trials, for the first time or not, which has (effectively) been given also by G. B. Price [Amer. Math. Monthly **53**, 59–74 (1946); these Rev. **7**, 309]. Finally, the simplifications for the special case of equal probabilities are noticed.

J. Riordan (New York, N. Y.).

Kappos, Demetrios A. Die Totaladditivität der Wahrscheinlichkeit. Bull. Soc. Math. Grèce **28**, 63–80 (1954).

A discussion of the various methods of extending a finitely additive probability measure on a Boolean algebra to a countably additive probability measure on a larger Boolean algebra.

P. R. Halmos (Chicago, Ill.).

Vorob'ev, N. N. Addition of independent random variables on finite abelian groups. Mat. Sbornik N.S. **34**(76), 89–126 (1954). (Russian)

The author studies the addition of independent random variables on finite abelian groups. He develops an apparatus of characteristic functions analogous to the classical one for real random variables and derives, among other things, necessary and sufficient conditions for the stability and infinite divisibility of distributions. Previous results on convergence to the uniform distribution on cyclical groups due to P. Lévy [Bull. Soc. Math. France **67**, 1–41 (1939); these Rev. **1**, 62] and A. Dvoretzky and J. Wolfowitz [Duke Math. J. **18**, 501–507 (1951); these Rev. **12**, 839] are included as very special cases. [However, the question of the rate of convergence is not considered in the present paper.]

A. Dvoretzky (New York, N. Y.).

Fisz, M. The limiting distributions of sums of arbitrary independent and equally distributed r-point random variables. Studia Math. **14** (1953), 111–123 (1954).

Proofs of results announced earlier [Bull. Acad. Polon. Sci. Cl. III. **1**, 235–238 (1953); these Rev. **15**, 635].

J. Wolfowitz (Ithaca, N. Y.).

Blackwell, David. On optimal systems. Ann. Math. Statistics **25**, 394–397 (1954).

L'auteur montre que, pour une suite de variables aléatoires $X_1, X_2, \cdots, X_n, \cdots$ telles que $|X_n| \leqq 1$,

$$E(X_n | X_1, \cdots, X_{n-1}) \leqq -u(\max |X_n| X_1, \cdots, X_n),$$

où u est une constante déterminée telle que $0 < u < 1$, on a pour tout $t > 0$:

$$\Pr[\sup_n (X_1 + \cdots + X_n) \geqq t] \leqq \left(\frac{1-u}{1+u}\right)^t,$$

cette inégalité étant la meilleure possible sous les hypothèses faites. Ce résultat, à rapprocher d'une théorème de S. Bernstein [cf. J. V. Uspensky, Introduction to mathe-

matical probability, McGraw-Hill, New York, 1937, p. 204], permet de préciser le théorème de P. Levy selon lequel, si

$$E(X_n|X_1, \cdots, X_{n-1})=0,$$

$$\Pr\left[\lim_{n\to+\infty} (X_1+\cdots+X_n)/n=0\right]=1.$$

R. Fortet (Paris).

Nash, Stanley W. An extension of the Borel-Cantelli lemma. Ann. Math. Statistics 25, 165–167 (1954).

Let $A_1, \cdots, A_n, \cdots$ be a sequence of not necessarily independent events. Then a necessary and sufficient condition that $P(\limsup A_n)=1$ is that for every sequence $B_1, \cdots, B_n, \cdots$ such that each B_i is equal to either A_i or the complement of A_i, only a finite number of B_i are the corresponding A's, and $P(\bigcap_{i=1}^{n} B_i)\neq 0$ for all n, $\sum_{j=1}^{\infty} P(A_j|\bigcap_{i=1}^{j} B_i)$ is infinite. *H. Rubin.*

Ryll-Nardzewski, C. On the non-homogeneous Poisson process. I. Studia Math. 14 (1953), 124–128 (1954).

The author defines a probability measure and stochastic processes on a suitable space of set functions in order to treat Poisson processes without imposing a homogeneity condition. *E. Lukacs* (Washington, D. C.).

Kuznecov, P. I., Stratonovič, R. L., and Tihonov, V. I. Quasi-moment functions in the theory of random processes. Doklady Akad. Nauk SSSR (N.S.) 94, 615–618 (1954). (Russian)

Let $t_1, \cdots, t_n$ be n time-points of the range of a random process $\xi(t)$. Denote by

$$f_n(u_1, \cdots, u_n; t_1, \cdots, t_n)=E\left\{\exp\left[i\sum_1^n u_\alpha \xi(t_\alpha)\right]\right\}$$

the characteristic function of the joint distribution of the random variables $\xi(t_1), \cdots, \xi(t_n)$. The quasi-moment functions $b_p(t_{\alpha_1}, \cdots, t_{\alpha_p})$ are defined by the relation

$$f_n(u_1, \cdots, u_n; t_1, \cdots, t_n)$$
$$\times\exp\left\{-i\sum_{\alpha=1}^{n} s(t_\alpha)u_\alpha-\frac{i^2}{2}\sum_{\alpha_1,\alpha_2=1}^{n} r(t_{\alpha_1}, t_{\alpha_2})u_{\alpha_1}u_{\alpha_2}\right\}$$
$$=\sum_{p=0}^{\infty}\frac{i^p}{p!}\sum_{\alpha_1,\cdots,\alpha_p=1}^{n} b_p(t_{\alpha_1}, \cdots, t_{\alpha_p})u_{\alpha_1}\cdots u_{\alpha_p}$$
with $b_0=1$.

Here $s(t)$ and $r(t_1, t_2)$ are given functions. The author expresses the quasi-moment functions in terms of the cumulant functions and extends his definition to the more general situation of two correlated random processes. The transformation of quasi-moment functions and their use in connection with stochastic differential equations is briefly discussed. *E. Lukacs* (Washington, D. C.).

Have

***Blanc-Lapierre, A., et Fortet, Robert. Théorie des fonctions aléatoires. Applications à divers phénomènes de fluctuation. Avec un chapitre sur la mécanique des fluides par J. Kampé de Fériet.** Masson et C^ie^, Paris, 1953. xvi+693 pp. Broché 6000 francs; cartonné 6500 francs.

This book, "for the usage of physicists", is intended to bridge the gap between the mathematical theory of stochastic processes and its applications. Markov processes (including Brownian motion and Poisson processes) and processes of the second order (including stationary ones) are the main topics. There is an adequate introduction going as far as the measure-theoretic foundations. Concrete models are presented together with formal discussions; intuitive and operative aspects are stressed; numerical examples are worked out. Some proofs are given, others not. There are many diagrams, figures and summaries.

If this book will impress upon the physicist the advantage of a little mathematical theory it will have served a good purpose. The mathematician will enjoy seeing abundant applications, although a honest one must recognize once more the futility of much beautiful theory. (At the same time it is well known that theory is powerless to cope with some of the "simplest" practical problems that arise.) While no new results are claimed quite a few topics are relatively unknown and up to now to be found only in original papers. The choice of material is necessarily subjective to a certain extent, especially when there is a competition for space between statements and proofs. Occasionally it may be debated whether a simplified version with more details would not be more helpful to the reader who wants to get a "feeling" for the subject. It is amusing, for instance, to find a sophisticated discussion of continuous-parameter ("permanent") Markov chains (denumerable) à la Doob, who did not choose to include it in his own book. Ch. 6, Sec. 11, on Markov chains of infinite order, seems another example of choice. The material is still pretty raw, and, as presented, this reviewer found it hard to digest. On the other hand, some crucial points are passed over, e. g., formula (6-4-8) on p. 204 which is the eigenvalue expansion of Markov matrices. Since this is made the basis of treatment of the finite case and it is the kind of operative formula which physicists rather cherish, a brief derivation should not be out of place. The usual reference to Fréchet's book does not help, since there the proof is first deferred to three Notes and then again to a paper in a (usually) inaccessible journal. In Feller's book only the case of simple roots is treated.

While the style is locally clear it tends to be somewhat diffuse at large. Perhaps some of it is inevitable because of the interpretative nature of the work. The reading is made harder, however, for the casual reader due to insufficient organization. For example, the new notation a_v is explained in a two-asterisk section ("to be omitted . . .") on p. 365 but it is widely used subsequently.

The comment on p. 230 on Doeblin's work is uncalled for. The first part of the Theorem on p. 264 is incorrect. Doob (1942) originally proved that if $X(t)=i$ then $|X(\tau)|$ as $\tau\to t$ may have at most two limiting values $+\infty$ and i. It can be shown under H_a that if $p_i=+\infty$ both are actual limiting values. (Lévy (1952) calls such a state i instantaneous in a process of the 5th type.)

Table of contents: I. Physical introduction to random functions. II. Axioms, notions, and main theorems of probability theory. III. Generalities on random functions. IV. Generalities on stochastic processes; random functions with independent increments. V. Random functions from Poisson processes. VI. Generalities on Markov processes. Homogeneous, discrete, Markov chains, finite case. Homogeneous, permanent, Markov chains, finite case. Homogeneous, permanent, Markov chains, denumerable case. VII. Permanent discontinuous Markov chains. Permanent continuous Markov chains. Additive functionals of a Markov process. VIII. Harmonic analysis of random functions; linear filters. IX. Energetic properties of second-order random functions. X. Some problems related to harmonic analysis. XI. Stationary random functions of second order. XII. Laplacian random functions. XIII. Application: noise and telecom-

munications. XIV. Random functions and the statistical theory of turbulence. XV. Mathematical appendix.

K. L. Chung (Syracuse, N. Y.).

Picard, H. C. Two random movements in a plane. Simon Stevin 30, 25–43 (1954). (Dutch. English summary)

The author evaluates the dispersion of the distance from the origin of a particle in a plane random walk, after n steps, and related quantities. For example, one case considered has the particle proceeding in straight line segments. The vector displacements are mutually independent with a specified common distribution.

J. L. Doob.

Goldberg, Samuel. Probability models in biology and engineering. J. Soc. Indust. Appl. Math. 2, 10–19 (1954).

Mathematical Statistics

Have

✱Probability tables for the analysis of extreme-value data. National Bureau of Standards Applied Mathematics Series No. 22. U. S. Government Printing Office, Washington, D. C., 1953. iii+32 pp. 25 cents.

This collection of tables is intended for the analysis of extreme observations, and some of the tables have already been published in journals as parts of papers by E. J. Gumbel and others. A summary of the theoretical background is as follows. Consider a variate x with range $a \leqq x < \infty$ (where a may be finite or $-\infty$), density function $f(x)$ and cumulative distribution $F(x)=\int_a^x f(t)dt$. (It is not specifically stated that it must be assumed that $f(x)>0$ for all large x.) Introduce further the root u of $F(u)=1-1/n$ and $\alpha=nf(u)$. Let x_m denote the mth largest observation (mth order statistic) in a sample of n drawn from this population; then the asymptotic cumulative distribution of x_n can be shown to be given by

$$\Pr\{x_n \leqq x\} \cong \Phi(y)=\exp\{e^{-y}\} \quad \text{with} \quad y=\alpha(x-u).$$

Likewise for $(n-m) \ll n$,

$$\Pr\{x_m \leqq x\} \cong \Phi_m(y_m)=\int_{-\infty}^{y_m} \frac{m^m}{(m-1)!} \exp\{-my-me^{-y}\}dy$$

with $y_m=\alpha_m(x-u_m)$, where u_m is the root of $F(u_m)=1-(m/n)$ and $\alpha_m=nf(u_m)/m$. As for large samples x_n and x_1 are approximately independent, the asymptotic distribution of range $w=x_n-x_1$ can, for infinite range symmetrical distributions, be obtained as that of the difference of two independent extremals, x_n and x_1, resulting in

$$\Pr\{w \leqq W\} \cong \Psi(R)=2e^{-\frac{1}{2}R}K_1(2e^{-\frac{1}{2}R})$$

where the 'reduced range' limit is defined as $R=\alpha(W-2u)$, the consequential density function of w is given by

$$\psi(R)=2e^{-R}K_0(2e^{-\frac{1}{2}R})$$

and K_0, K_1 are the modified Bessel functions of the first and second kind. When, as is usually the case, the population parameters u, α or u_m, α_m are not known, they can be estimated from a sample of N observed x_n or w values (by methods which are discussed).

Table 1 gives

$$\Phi(y)=\exp(-e^{-y}) \quad \text{and} \quad \phi=d\Phi/dy=\exp(-y-e^{-y})$$

to 7 decimals for $y=-3.0(0.1)-2.4(.05)0(.1)4(.2)8(.5)17$; second differences (sometimes modified) are given. Table 2 gives the inverse $y=-\log_e(-\log_e \Phi)$ to 5 decimals for $\Phi=.0001(.0001).005(.001).988(.0001).9994(.00001).99999$.

Table 3 gives ϕ to 5 decimals as a function of Φ for

$$\Phi=0(.0001).01(.001).999.$$

Table 4 gives the 'percentage points' of y_m, i.e., the roots y_m of $\Phi_m=\Phi_m(y_m)$ to 5 decimals for $\Phi_m=.005$, .01, .025, .05, .1, .25, .5, .995, .99, .975, .95, .9, .75 and $m=1(1)15(5)50$. Table 5 gives $\Psi(R)$ and $\psi(R)$ to 7 decimals for

$$R=-4.6(.1)-3.3(.05)11(.5)20;$$

δ^2 (sometimes modified) is tabled. Table 6 gives the inverse to Table 5, i.e., $R=R(\Psi)$ to 3 decimals for

$$\Psi=.0001(.0001).001(.001).01(.01).95(.001).999(.0001).9999.$$

H. O. Hartley (Ames, Iowa).

✱Monjallon, Albert. Introduction à la méthode statistique. Librairie Vuibert, Paris, 1954. 279 pp. 2000 francs.

Have

Shenton, L. R. Inequalities for the normal integral including a new continued fraction. Biometrika 41, 177–189 (1954).

The author gives continued-fraction expansions for Mill's ratio $R(t)=\exp\{\frac{1}{2}t^2\}\int_t^\infty \exp\{-\frac{1}{2}x^2\}dx$ and for the related ratio $\bar{R}(t)=\exp\{\frac{1}{2}t^2\}\int_0^t \exp\{-\frac{1}{2}x^2\}dx$. These expansions are used to derive two sets of inequalities for $R(t)$ which generalize results obtained by Z. W. Birnbaum [Am. Math. Statistics 13, 245–246 (1942); these Rev. 4, 19].

E. Lukacs (Washington, D. C.).

Irwin, J. O. A distribution arising in the study of infectious diseases. Biometrika 41, 266–268 (1954).

The author describes a joint distribution for the chance variables $X_1, X_2, \cdots, X_n$ as follows. X_i can be either zero or one, and $P(X_i=1)=p$, for $i=1, \cdots, n$. Also, the correlation coefficient between X_i and X_j is r, for i, $j=1, \cdots, n$ $(i \neq j)$. p and r are given numbers. Then the author sets out to find "the" distribution of the chance variable X defined as $X_1+X_2+\cdots+X_n$. What he actually finds is one of the many possible distributions of X, for as the simple example in the next paragraph will show, the description given of the joint distribution of $X_1, X_2, \cdots, X_n$ does not uniquely determine a distribution for X. The author's equation (1), which led to "the" distribution of X, really introduces additional assumptions about the joint distribution of $X_1, X_2, \cdots, X_n$.

Let G be any number in the open interval $(2/32, 5/32)$. Suppose $P(X_1=u, X_2=v, X_3=w)$ is equal to G if u, v, w are all zero, is equal to $5/32-G$ if exactly one of the values u, v, w equals unity, is equal to $G-2/32$ if exactly two of the values u, v, w are equal to unity, and is equal to $23/32-G$ if all three of the values u, v, w are equal to unity. Then $P(X_i=1)=3/4$ for $i=1, 2, 3$; while the correlation coefficient between any two of our three chance variables is $\frac{1}{3}$. But $P(X=0)=G$, $P(X=1)=15/32-3G$, $P(X=2)=3G-6/32$, $P(X=3)=23/32-G$, thus giving a whole family of possible distributions for X.

L. Weiss (Charlottesville, Va.).

Box, G. E. P. Some theorems on quadratic forms applied in the study of analysis of variance problems. I. Effect of inequality of variance in the one-way classification. Ann. Math. Statistics 25, 290–302 (1954).

The author derives the exact distribution of a weighted sum of independent chi-squares with even degrees of freedom

and the exact distribution of the ratio of independent sums of this type. He tests numerically simple approximations to these distributions. The results are used to determine the effect of variance-heterogeneity on the distribution of F in the one-way classification. The largest deviations are found when the groups are unequal. (The inequality within brackets in formula (2.11) should read $\lambda_f\chi^2(2s) > X_0$.)

D. M. Sandelius (Göteborg).

Gurland, John. Distribution of quadratic forms and ratios of quadratic forms. Ann. Math. Statistics 24, 416–427 (1953).

The problems of finding the distribution of any definite quadratic form, and the distribution of the ratio of any indefinite quadratic form to any nonnegative quadratic form XPX' are solved by means of a Laguerrian expansion due to Szegö [Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939; these Rev. 1, 14], where $X=(X_1, \cdots, X_n)$ is distributed in an n-dimensional normal distribution. The method appeals to the calculation of the semimoments, which is rather difficult to obtain except in the case where the distribution of an indefinite quadratic form is symmetric. The author proposes also a new system of orthogonal polynomials, which obviates the need of semimoments. *T. Kitagawa* (Fukuoka).

Kondo, Takayuki. Evaluation of some ω_n^2 distribution. J. Gakugei, Tokushima Univ. (Nat. Sci.) 4, 45–47 (1954).

Watanabe [J. Gakugei Coll. Tokushima Univ. 2, 21–30 (1952); these Rev. 14, 775] gave the distribution ω_n^2. Here ω_n^2 is a statistic used to test goodness of fit; it is the form Smirnov's modification of von Mises ω^2 statistic takes when the sample observations are grouped (for the exact definition see the above reference). In the present article the ω_n^2 distribution is tabulated for $n=2$ and $n=9$. Some discussion is given of the numerical computation methods used.

D. G. Chapman (Seattle, Wash.).

Silvey, Samuel D. The asymptotic distributions of statistics arising in certain non-parametric tests. Proc. Glasgow Math. Assoc. 2, 47–51 (1954).

Sufficient conditions are given for the joint asymptotic normality of p linear forms under randomization. The asymptotic distribution under randomization of the analysis of variance statistic is obtained. The problem of m-rankings is discussed. *G. E. Noether* (Boston, Mass.).

Lord, R. D. The use of the Hankel transform in statistics. I. General theory and examples. Biometrika 41, 44–55 (1954).

The use of the characteristic function in problems of the addition of independent random vectors is well established. This paper is concerned with the particular forms taken when the vectors have spherical distributions in s dimensions, i.e., when all directions are equally probable and the distribution of magnitudes is independent of direction. That the characteristic function is then a Hankel transform follows by changing to polar co-ordinates in the usual Cartesian form. A change to polars has sometimes been used in particular problems of this class but usually in a manner which does not show that the Hankel transform is the natural tool and that there is a theory for spherical distributions which runs parallel to the theory of the addition of random variables in one dimension. (From the author's introduction.) *J. Wolfowitz* (Ithaca, N. Y.).

Rényi, Alfréd. On the theory of order statistics. Acta Math. Acad. Sci. Hungar. 4, 191–231 (1953). (Russian summary)

English version of the author's paper in Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 467–503 (1953); these Rev. 15, 544.

Dunnett, Charles W., and Sobel, Milton. A bivariate generalization of Student's t-distribution, with tables for certain special cases. Biometrika 41, 153–169 (1954).

Suppose z_1, z_2 have a nondegenerate bivariate normal distribution with zero means, common variance σ^2, and correlation coefficient ρ, while ns^2/σ^2 has a chi-square distribution with n degrees of freedom and is distributed independently of z_1 and z_2. Define t_1 as z_1/s, t_2 as z_2/s, and $P_n(h, k; \rho)$ as $P(t_1 \leq h, t_2 \leq k)$. An exact expression for $P_n(h, k; \rho)$ is found, but it becomes increasingly cumbersome as n increases. An asymptotic series for $P_n(h, k; \rho)$ in powers of n^{-1} is given. Given a number P in the open interval $(0, 1)$, $t_n(P; \rho)$ is defined as the value of t such that $P_n(t, t; \rho) = P$. An asymptotic expression for $t_n(P; \rho)$ in powers of n^{-1} is given. Values of $P_n(t, t; \rho)$ are tabulated for $\rho = \pm .5$ and various values of n and t. Values of $t_n(P; \rho)$ are tabulated for $\rho = \pm .5$ and various values of n and P. These tables are needed to carry out the decision procedure described in the paper reviewed below. *L. Weiss* (Charlottesville, Va.).

Bechhofer, Robert E., Dunnett, Charles W., and Sobel, Milton. A two-sample multiple decision procedure for ranking means of normal populations with a common unknown variance. Biometrika 41, 170–176 (1954).

Let X_{ij} be normally and independently distributed chance variables from population π_i with mean μ_i and variance $\sigma_i^2 = a_i\sigma^2$ $(i=1, \cdots, k;\ j=1, 2, \cdots)$. σ^2 and the μ_i are unknown, the a_i are known positive integers. Two separate problems are considered. In the first, the population having the largest mean is to be chosen, by a procedure which guarantees a probability of at least P (a preassigned number) of choosing the correct population as long as the difference between the largest population mean and the next largest is at least a given preassigned number. The second problem is to rank all k populations according to the size of their means, by a procedure which guarantees at least a preassigned probability P of a correct ranking as long as the $k-1$ positive differences between adjoining ordered population means are at least as great as given preassigned values. No single sample plan will solve either of these problems, and the authors construct a procedure analogous to Stein's two-sample procedure [Ann. Math. Statistics 16, 243–258 (1945); these Rev. 7, 213]. First, a sample of size a_iN_0 is taken from π_i $(i=1, \cdots, k)$. From this, the authors construct a certain estimate of σ^2, and the number of additional observations to be taken from π_i is a certain nondecreasing function of this estimate, for $i=1, \cdots, k$. Finally, the overall mean of all the observations from π_i is computed for $i=1, \cdots, k$, and the populations are ranked according to the ranking of these observed overall means. A formula for the expected sample size is given. *L. Weiss.*

Olkin, I., and Roy, S. N. On multivariate distribution theory. Ann. Math. Statistics 25, 329–339 (1954).

The authors suggest "a matrix method of deriving the sampling distributions of a large class of statistics directly from the probability law for random samples from a multivariate normal population, that is without assuming the Wishart distribution or the distribution of rectangular co-

ordinates". (They remark that the Wishart distribution is unavailable in the case of a sample of N observations from a p-variate normal population with $p>N-1$.) In most cases the distribution of the (dependent) variables in question can be obtained by "(a) finding an appropriate transformation [of the original variates], (b) evaluating the Jacobian of the transformation, and (c) integrating out any extraneous variates. This paper is concerned with (b) and (c); the requisite transformations (a) are assumed to be available." The transformation may involve an orthogonal matrix. The authors propose two procedures for dealing with the constraints imposed by orthogonality: one uses a theorem on Jacobians [cf. S. N. Roy, Calcutta Statist. Assoc. Bull. 4, 117–122 (1952); these Rev. 14, 959], and the other involves the introduction of extra variates. They illustrate the first procedure by deriving the distribution of the rectangular coordinates, and the second by deriving that of the roots of a certain determinantal equation. Both these distributions were known [cf. Mahalanobis, Bose, and Roy, Sankhyā 3, 1–40 (1937); R. A. Fisher, Ann. Eugenics 9, 238–249 (1939); these Rev. 1, 248]. A note added in proof refers to a similar result obtained by different methods by A. T. James [Ann. Math. Statistics 25, 40–75 (1954); these Rev. 15, 726]. *H. P. Mulholland.*

Kitagawa, Tosio. The *t*-distributions concerning random integrations. Mem. Fac. Sci. Kyūsyū Univ. A. 8, 31–41 (1953).

In a previous paper [Bull. Math. Statist. 4, 15–21 (1950); these Rev. 14, 457] the author introduced three types of random approximation to the integral of a continuous function $f(t)$ over the interval $(0, 1)$, and found their mean values and variances. He now investigates the corresponding t-distributions. In particular, for the first type, where $f(t)$ is a fixed function, if we use $\bar{f}=n^{-1}\sum_{i=1}^{n}f(t_i)$ as the approximation, $t_1, \cdots, t_n$ being independent random variables uniformly distributed on $[0, 1]$, and write

$$s^2=(n-1)^{-1}\sum\{f(t_i)-\bar{f}\}^2,$$

the distribution of $t=n^{\frac{1}{2}}\bar{f}/s$ is required. For the other two types a modified form of t is considered. Under certain restrictive assumptions the author obtains the ordinary t-distribution as a first approximation to the distribution required, and uses certain results, involving Gram-Charlier Type A expansions, due to C.-E. Quensel [Skand. Aktuarietidskr. 26, 210–219 (1943); these Rev. 7, 212] and H. Uranisi [Bull. Math. Statist. 4, 1–14 (1950); these Rev. 14, 63] to obtain better approximations. The formulae involved are too elaborate to be reproduced here.

H. P. Mulholland (Birmingham).

Kitagawa, Tosio. Some stochastic considerations upon empirical functions of various types. Bull. Math. Statist. 5, no. 3–4, 19–33 (1953).

The author considers the estimation of a function $g(t)$, $0\leq t\leq 1$, that is either (A) fixed, or (B) a sample function of a certain stochastic process, and whose values are observed either (C) continuously throughout $[0, 1]$ or (D) at the points of a fixed finite set. For the cases (AD), (BD), (AC), and (BC) he endeavours to obtain generalizations of the distribution of "Student's" ratio. For instance, in the case (AC), let $g(t)$ have the generalized Fourier expansion $\sum_{\nu=1}^{\infty} a_\nu\lambda_\nu^{-\frac{1}{2}}\phi_\nu(t)$, where $\{\phi_\nu(t)\}$ and $\{\lambda_\nu\}$ are characteristic functions and values of an integral equation $\phi(t)=\lambda\int_0^1 r(t, s)\phi(s)ds$ and $\{\phi_\nu(t)\}$ is a complete orthonormal system in $L^2(0, 1)$. For each i ($i=1, \cdots, m$) let the a_ν's be replaced by random variables $\alpha_{i\nu}$ with normal distributions $N(a_\nu, \sigma_1^2)$, all the $\alpha_{i\nu}$'s being independent: we thus get, instead of $g(t)$, m independent random functions $f_i(t)$. The author estimates $g(t)$ by $\bar{f}(t)=\sum_{i=1}^{m} f_i(t)$, writes $s^2(t)=(m-1)^{-1}\sum_{i=1}^{m} \{f_i(t)-\bar{f}(t)\}^2$ and obtains (Theorem 1.4) the probability density for the distribution of

$$\int_0^1\{\bar{f}(t)-g(t)\}^2dt \Big/ \int_0^1 s^2(t)ds$$

in the form of a single integral of a certain (complicated) algebraic function: he uses a method due to M. Kac [Proc. 2nd Berkeley Symposium Math. Statistics and Probability, 1950, Univ. of California Press, 1951, pp. 189–215, especially §1; these Rev. 13, 568]. In the cases (AD), (BC), and (BD) he only initiates the calculation of the requisite distributions. The treatment of (AD) seems rather confused. An estimate $\hat{g}_m(t)$ for $g(t)$ is formed from means that are treated as of equal weight, though derived from unequal numbers of observations; in (1.08), p. 22, $g(t)-\hat{g}_m(t)$ should be $g_m(t)-g(t)$, where $g_m(t)$ is what $\hat{g}_m(t)$ becomes when all errors vanish; and in (1.26) and (1.28) of Theorem 1.1, p. 23, $R-1$ should be replaced by R. [In (AD) the reviewer does not see that the author's approach has any advantage over the more usual treatment of curvilinear regression by analysis of variance.]

The second part of the paper indicates some corresponding results for estimates of the derivative $g'(t)$.

H. P. Mulholland (Birmingham).

Mittmann, Otfrid M. J. Betrachtungen zur Analyse empirischer Funktionen. Z. Angew. Math. Mech. 34, 37–43 (1954). (Russian summary)

The author treats the problem of estimating the mean value curve of a Gaussian process on the basis of one observed sample curve. He chooses a system of functions $\{\phi_s(x)\}$, divides the range into sections and approximates the sample curve in each section by a sum $\sum_{s=1}^{n} a_s\phi_s(x)$. A rather crude method is proposed to decompose each coefficient a_s into a "random component" $\tilde{a}_s$ and a "systematic component" $\bar{\bar{a}}_s$. The sum $\sum_{s=1}^{n}\bar{\bar{a}}_s\varphi_s(x)$ is then the desired estimate. No use is made of the available tools from the theory of stochastic processes or the theory of statistical estimation. *E. Lukacs* (Washington, D. C.).

Jonckheere, A. R. A distribution-free *k*-sample test against ordered alternatives. Biometrika 41, 133–145 (1954).

A test of the hypothesis that k samples have come from the same population against the alternative that $F_1(x)<F_2(x)<\cdots<F_k(x)$ is investigated. The test statistic S is based on the number of times an observation from the ith population is smaller than an observation from the jth population, $i<j$. The exact distribution of S is tabulated for $k=3(1)6$ in the case that the k samples are of equal size (≤ 5). The asymptotic distribution of S is also discussed.

G. E. Noether (Boston, Mass.).

van Eeden, Constance. Methods for comparing, testing and estimating unknown probabilities. Statistica, Rijswijk 7, 141–162 (1953). (Dutch. English summary)

A superficial exposition of statistical tests and the related confidence intervals for a 2×2 contingency table with fixed marginal totals. *H. L. Seal* (New York, N. Y.).

Gurland, John. An example of autocorrelated disturbances in linear regression. Econometrica **22**, 218–227 (1954).

Critical comments on the shortcut suggested by Cochrane and Orcutt [J. Amer. Statist. Assoc. **44**, 32–61 (1949)] for deriving best linear unbiased estimates of regression parameters. *H. Wold* (Uppsala).

Spetner, Lee M. Errors in power spectra due to finite sample. J. Appl. Phys. **25**, 653–659 (1954).

The author considers the error produced in the estimate of the power spectrum of a continuous stationary series by the fact that calculations are based on a finite sample. He derives and tabulates useful upper bounds for the expected bias of the two components of a periodogram ordinate.

The statement is made that smoothing of the spectrum by averaging of neighbouring ordinates is equivalent to smoothing by averaging spectra of distinct sections of the series. This is not correct, although the two averages will have approximately the same distribution if the spectrum varies but slowly. *P. Whittle* (Wellington).

Middleton, D., Peterson, W. W., and Birdsall, T. G. Discussion of "Statistical criteria for the detection of pulsed carriers in noise I, II". J. Appl. Phys. **25**, 128–130 (1954).

The senior author has given the first unified presentation of the application of several types of statistical tests to the problem of detection in communication theory, and this note contains a clearer and more complete account of the author's method of comparison than that given in same J. **24**, 379–391 (1953) [these Rev. **14**, 1105]. *S. Ikehara.*

Laurent-Duhamel, Marie-Jeanne. Étude statistique de contours craniens considérés comme courbes aléatoires. Publ. Inst. Statist. Univ. Paris **2**, no. 3, 27–54 (1953).

For each of the heads of 100 men and for each of 100 skulls from a museum collection the polar equation to the curve of intersection with a standard horizontal plane has been determined with the aid of an instrument designed for use by hatters. The author presents in graphical form a selection of the results and tabulates in full the first 24 Fourier coefficients for each of the 200 contours (taken in the polar form $\rho=f(\omega)$). Several descriptive statistics are tabulated and the material is discussed from the standpoint of recent work by M. Fréchet [see, e.g., ch. VI of his book "Généralités sur les probabilités. Éléments aléatoires", 2e éd., Gauthier-Villars, Paris, 1950; these Rev. **12**, 423]. *D. G. Kendall* (Oxford).

Mathematical Economics

***Bellman, Richard. An introduction to the theory of dynamic programming.** The Rand Corporation, Santa Monica, Calif., 1953. x+99 pp.

The author begins with a list of seven problems which can be attacked by solving a functional equation of the form

$$(1)\qquad f(p)=\max_k T_k f(p),$$

where $f(p)$ is a real-valued function defined on a space P and each T_k is an operator mapping functions on P into functions on P. Generally speaking, (1) arises in the following context. There is a system whose set of states is P, and a set K of operations k, each of which, applied to the system, transforms the initial state into a new state according to a specified (deterministic or probabilistic) law. A sequence of operations, applied successively, produces a sequence of states, and there is associated with each possible pair of sequences a value. The problem is to determine sequences of the greatest value; if $f(p)$ is the value of the best possible sequence when the initial state is p, and $T_k f(p)$ is the value of the best possible sequence if in addition the initial operation is k, then $f(p)$ satisfies (1). The point is that the operator T_k is often of a simple type.

In chapter 2, several existence and uniqueness theorems are proved for special cases of (1). Later chapters investigate in more detail the solutions of the special cases of (1) corresponding to certain particular problems.

A typical problem treated is stated as follows: "We are informed that a particle is in either state 0 or 1, and we are given initially the probability x that it is in state 1. Use of the operation A will reduce this probability to ax, where a is some positive constant less than 1, whereas operation L, which consists in observing the particle, will tell us definitely which state it is in. If it is desired to transform the particle into state 0 in a minimum time, what is the optimal procedure?" For this problem, if $f(x)$ is the expected time consumed in an optimum procedure, then

$$(2)\quad f(x)=\min\,[1+xf(1),\,1+f(ax)],\quad x>0,\quad f(0)=0.$$

One of the existence and uniqueness theorems implies that (2) has a unique positive bounded solution. This solution is given by $f(1)=\min_{k=1,2,\cdots}(k+1)/(1-a^k)$ and, writing $x_0=[(1-a)f(1)]^{-1}$,

$$\begin{aligned} f(x)&=1+xf(1), && x\leqq x_0\\ &=(N+1)+a^N xf(1), && ax_0\leqq a^N x\leqq x_0,\quad N=1,2,\cdots. \end{aligned}$$

D. Blackwell (Washington, D. C.).

Bellman, Richard. Some functional equations in the theory of dynamic programming. Proc. Nat. Acad. Sci. U. S. A. **39**, 1077–1082 (1953).

Solutions are given for some functional equations of the form (1) of the preceding review. *D. Blackwell.*

Shapley, L. S. Stochastic games. Proc. Nat. Acad. Sci. U. S. A. **39**, 1095–1100 (1953).

The author considers an interesting class of two-person zero-sum multi-stage games. These lead, when formulated in terms of behavior strategies, to functional equations of the form, $\phi_k=\mathrm{val}\,[A_k(\phi)]$, $k=1,2,\cdots,N$, where $\mathrm{val}\,[A_k(\phi)]$ denotes the value of the game whose matrix is $A_k(\phi)$, and ϕ is a vector whose components are ϕ_k. In an ingenious manner, the author shows that these equations may be solved explicitly in the form

$$\phi_k=\min_x\max_y R_k(x,y)=\max_y\min_x R_k(x,y),$$

where $R_k(x,y)$ is a rational function determined by the solution of the linear equations associated with the functional equation above. By a proper choice of matrices, this result yields a known result of von Neumann concerning the minimax of the rational form $(xAy)/(xSy)$, where S is a positive matrix.

As the author points out, one-person versions of these functional equations occur in the theory of dynamic programming [cf. R. Bellman, same Proc. **38**, 716–719 (1952); these Rev. **14**, 392; and also the paper reviewed above, where "games of survival" are discussed]. *R. Bellman.*

Bellman, Richard. Bottleneck problems and dynamic programming. Proc. Nat. Acad. Sci. U. S. A. **39**, 947–951 (1953).

This paper indicates an application of the author's theory of "dynamic programming" [same Proc. **38**, 716–719 (1952); these Rev. **14**, 392] to a class of problems, of which the following "bottleneck problem" is asserted to be typical: Maximize $x_2(T)$ subject to

$$dx_1/dt = a_1y_1, \quad x_1(0) = c_1,$$
$$dx_2/dt = -y_1 - y_2 + \min(b_1x_1, a_2y_2) = (a_2-1)y_2 - y_1,$$

and $x_2(0) = c_2$, where $y_1, y_2 \geqq 0$, $y_1 + y_2 \leqq x_2$, and $y_2 \leqq b_2x_1/a_1$. It should be noted that the roles of x_1 and x_2 seem to be reversed in the interpretation. A complete treatment is promised. *H. W. Kuhn* (Bryn Mawr, Pa.).

Nikaidô, Hukukane. Note on the general economic equilibrium for nonlinear production functions. Econometrica **22**, 49–53 (1954).

Suppose there exist n well defined commodities. A production function is a single-valued function from n-dimensional Euclidean space to itself. Suppose the possible inputs form a bounded closed convex set in the positive orthants which does not contain the origin, that the production function is continuous, that each input plus the corresponding output is positive, and that the production function is convex. Then an economic equilibrium of the usual type exists.

H. Rubin (Stanford, Calif.).

Georgescu-Roegen, Nicholas. Note on the economic equilibrium for nonlinear models. Econometrica **22**, 54–57 (1954).

The author generalizes the results of the preceding paper as follows. First, instead of requiring a production function, the author introduces a technological horizon which consists of the set of all technologically feasible input-output pairs. He assumes, in addition, that the technological horizon forms a closed convex set in the positive orthant and that the total of the inputs and the sum of each input and the corresponding outputs are bounded away from zero. In addition he assumes that the shadow of the technological horizon on the probability simplex is closed. Under these conditions a technological equilibrium exists, but the technological equilibrium may not be achievable because the resources are insufficient. If the available inputs form a closed convex set with a closed normalized shadow then there is also an achievable equilibrium. *H. Rubin.*

May, Kenneth O. Intransitivity, utility, and the aggregation of preference patterns. Econometrica **22**, 1–13 (1954).

The author takes the point of view that theories of choice should be "built to describe behavior as it is rather than 'ought to be' ". He points out that, experimentally, we do not have transitivity of preference in the behavior of individuals, and for groups it was shown by K. J. Arrow [Social choice and individual values, Wiley, New York, 1951; these Rev. **12**, 624] that under certain reasonable conditions no group choice function could be constructed to satisfy the axioms which are usually given for individual choices. The author shows that if the transitivity axiom is eliminated, it is rather easy to construct group preferences which do not otherwise violate Arrow's conditions. It is to be pointed out that the author does not develop any theory of nontransitive preference comparable with the usual theory of transitive preference. *H. Rubin* (Stanford, Calif.).

Debreu, Gerard. A classical tax-subsidy problem. Econometrica **22**, 14–22 (1954).

The definition of the economic loss in a situation given by the author [Econometrica **19**, 273–292 (1951)] is applied to the introduction of a system of indirect taxes and subsidies into a Pareto-optimal state of the economy. The money value of this loss turns out to be, if the final state is close to the initial one,

$$-\tfrac{1}{2}[dt^+\cdot dq^+ - dt^-\cdot dq^- + d\xi(dp^- - dp)],$$

where all differentials without subscripts are vectors. For the hth commodity dt_h^+ is the amount of the tax (or subsidy) on a unit when it is an input (the superscript $+$ becomes $-$ for an output); dq_h^+ is the variation of the gross input of the hth commodity for the whole production sector due to the introduction of the tax-subsidy system. $d\xi$ is the excess of dx, the actual variation of total consumption, over Xdp, the variation which would result from the price change dp if all individual satisfactions were held constant. p^- is an intrinsic price vector defined in the paper cited above.

H. S. Houthakker (Stanford, Calif.).

TOPOLOGY

***Senior, James K. Unigraphic partitions. A simplified proof.** George Herbert Jones Laboratory, University of Chicago, 1954. 18 pp. 75 cents.

In how many ways can a graph be constructed which has a given number of vertices of each degree, the total number of vertices being finite? The author obtains necessary and sufficient conditions for the answer to be 0, or to be 1. His argument is an improved form of that used in an earlier paper [Amer. J. Math. **73**, 663–689 (1951); these Rev. **13**, 147]. *W. T. Tutte* (Toronto, Ont.).

Dirac, G. A. Theorems related to the four colour conjecture. J. London Math. Soc. **29**, 143–149 (1954).

The author considers an operation called "contraction" applied to a graph. An edge A, its ends a and b, and all other edges joining a and b are deleted. Then a new vertex is introduced to serve as an end of each remaining edge which was incident with a or b. It has been conjectured that each k-chromatic graph has a subgraph which can be contracted into a complete k-graph by a sequence (possibly empty) of such operations. The author proves the following weaker version of the conjecture: If a critical k-chromatic graph contains a complete n-graph as a subgraph, where $n \leqq k-1$, then the graph can be contracted into a complete $(n+1)$-graph. He uses this to show that any 6-chromatic graph drawn on a surface of characteristic 0 has a subgraph contractible into a complete 5-graph. He gives a new proof of the five-colour theorem for the sphere, depending on this result.

W. T. Tutte (Toronto, Ont.).

Hönig, Chaim Samuel. On a method of refinement of topologies. Bol. Soc. Mat. São Paulo **6** (1951), 1–52 (1952). (Portuguese)

This paper was reviewed as the author's thesis [São Paulo, 1952] in these Rev. **14**, 669.

Kurepa, Svetozar. Peano's transformations and Suslin's problem. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **8**, 175–190 (1953). (Serbo-Croatian. English summary)

Ein topologischer Raum E hat die Eigenschaft (S) (Suslinsche Eigenschaft), wenn jede Menge disjunkter Umgebungen in E höchstens abzählbar ist. Im Folgenden wird eine stetige geordnete Menge E als topologischer Raum betrachtet, indem man unter Umgebungen offene Intervalle versteht; E^2 soll das topologische Produkt $E\times E$ bedeuten. Verf. beweist folgende Sätze. I. Besitzt die stetige geordnete Menge E mit erstem und letztem Element die Eigenschaft (S), so sind die folgenden Behauptungen äquivalent: 1) E^2 hat die Eigenschaft (S); 2) E^2 ist ein stetiges Bild von E. II. Besitzt die stetige geordnete Menge E ohne erstes und letztes Element die Eigenschaft (S), so sind die folgenden Behauptungen äquivalent: 1) E ist dem Intervall $(0, 1)$ ähnlich; 2) es gibt eine die Rangordnung erhaltende Abbildung der Menge E in sich selbst, deren Diskontinuitätspunkte eine in E dichte Menge bilden. [Bemerkung des Referenten: In der Formel (4.1) soll 2^{-n} anstatt n^{-2} stehen.]
M. Novotný (Brno).

Knaster, B., et Reichbach, M. Notion d'homogénéité et prolongements des homéomorphies. Fund. Math. **40**, 180–193 (1953).

Let P, Q be homeomorphic subsets of metric spaces M, N, and suppose each of their complements $M-P$, $N-Q$, is the union of a null sequence of disjoint open-closed sets. The authors give conditions sufficient to ensure that the homeomorphism between P and Q can be extended to one between M and N. They make two main applications of this result. Firstly, if M and N are compact, perfect and totally disconnected, then any homeomorphism h between two closed subsets P, Q of M and N, which is such that $h(\mathrm{Fr}(P))=\mathrm{Fr}(Q)\neq 0$, can be extended to a homeomorphism between M and N. This theorem, which expresses a strong homogeneity property of the Cantor set, was conjectured by Knaster and first proved, in another way, by Ryll-Nardzewski [unpublished]. Secondly, if M, N are compact and have homeomorphic αth derived sets M^α, N^α (where α is any countable ordinal), and if $0\neq M^\alpha\subset \mathrm{Cl}(M^\xi-M^{\xi+1})$ for every $\xi<\alpha$, with a similar condition for N, then the homeomorphism between M^α and N^α can be extended to one between M and N. This generalizes the theorem of Mazurkiewicz and Sierpinski [Fund. Math. **1**, 17–27 (1920)] classifying the countable compacta. Various corollaries and related results are given. [There are some minor slips and misprints, a few of which may cause trouble on a first reading. The wording of Corollary 1 is misleading. The proof of Theorem 3, as given, assumes $h(P\cap \mathfrak{C}_1)=Q\cap \mathfrak{C}_2$, though the argument is easily adjusted. In Corollary 3, $\mathfrak{C}_1-B$ and $\mathfrak{C}_2-B$ should apparently be $\overline{\mathfrak{C}_1-B}$ and $\overline{\mathfrak{C}_2-B}$. In Corollary 5, the set H is presumably required to be compact.] *A. H. Stone.*

Brouwer, L. E. J. Ordnungswechsel in Bezug auf eine coupierbare geschlossene stetige Kurve. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 112–113 (1954).

In §4 of an earlier paper [Akad. Wetensch. Amsterdam, Proc. **28**, 503–508 (1925)] the author proved that for a Jordan curve two points P and Q can be found so that P is situated positively inside J and Q is situated positively outside Q. Here he proves more generally that for a closed continuous curve K which is "coupierbar" in a point S, two points can be found which are distant from K and have different orders with respect to K. *A. Heyting.*

Borsuk, K. On the decomposition of a locally connected compactum into Cartesian product of a curve and a manifold. Fund. Math. **40**, 140–159 (1953).

The principal result is that if X_1 and X_2 are locally connected compacta of dimension $\leqq 1$ and Y_1 and Y_2 are topological manifolds (with or without boundary but compact and connected) then homeomorphism of $X_1\times Y_1$ with $X_2\times Y_2$ and of Y_1 with Y_2 imply homeomorphism of X_1 with X_2. This may be contrasted with the known fact that homeomorphism of $X_1\times Y_1$ with $X_2\times Y_2$ and of X_1 with X_2 do not imply homeomorphism of Y_1 with Y_2. There is also deduced as corollary that decomposition of a space into a Cartesian product of a locally connected compactum of dimension $\leqq 1$ and of a finite number of simple arcs and simple closed curves is unique. Previous contributions to the problem of uniqueness of Cartesian factoring were noted in these Rev. **2**, 73; **8**, 48; **9**, 51; **10**, 316; **14**, 71; **15**, 337.
R. H. Fox (Princeton, N. J.).

Dowker, Yael Naim, and Friedlander, F. G. On limit sets in dynamical systems. Proc. London Math. Soc. (3) **4**, 168–176 (1954).

Let T be an automorphism of a compact metric space X. X is called T-connected if it contains no proper closed subset A such that $TA\subset \mathrm{int}\, A$. It is shown that (X, T) can be embedded in a larger system (Y, S) in such a way that X is the ω-limit set of some point of Y if and only if X is T-connected. Similarly for a continuous flow (X, T_t), X is the ω-limit set of a point of some flow (Y, S_t) containing (X, T_t) if and only if X is T_1-connected, or equivalently, if and only if X contains no proper closed subset A such that $T_tA\subset \mathrm{int}\, A$ for all sufficiently large t. In the discrete case, T-connectedness does not imply connectedness but is analogous at least to the extent that a T-connected space cannot be decomposed into α disjoint invariant closed sets where $2\leqq\alpha\leqq\aleph_0$. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Nemyckiĭ, V. V. Topological problems of the theory of dynamical systems. Amer. Math. Soc. Translation no. **103**, 85 pp. (1954).

Translated from Uspehi Matem. Nauk (N.S.) **4**, no. 6(34), 91–153 (1949); these Rev. **11**, 526.

Moise, Edwin E. Affine structures in 3-manifolds. VIII. Invariance of the knot-types; local tame imbedding. Ann. of Math. (2) **59**, 159–170 (1954).

[For parts I–VII see these Rev. **13**, 484, 574, 765; **14**, 72, 1003; **15**, 337.] Theorem (6.1). If some homeomorphism ϕ of 3-space E^3 upon itself preserves the orientation and transforms a given simple closed polygon J into a given simple closed polygon J', then there is a piecewise linear homeomorphism with these properties. (Actually the proof justifies the stronger statement in which J and J' are oriented and $\phi|J$ is required to be orientation-preserving.) In conjunction with results of Graeub [S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1950**, 205–272; these Rev. **13**, 152], this shows that the combinatorial knot-types (two simple closed polygons are of the same combinatorial type if a regular normalized projection of one of them can be transformed into a regular normalized projection of the other one by a finite sequence of "moves" Ω.1.2.3. [Reidemeister, Knotentheorie, Springer, Berlin, 1932, §3]) are in natural one-to-one correspondence with the topological knot-types.

(Two tame simple closed curves are of the same topological knot-type if one can be transformed into the other by an orientation-preserving autohomeomorphism of space.) Theorem (8.1): An imbedding of a set in a triangulated 3-manifold is tame if it is locally tame. Corollaries: Theorem (9.1): Every 3-manifold with boundary is triangulable. Theorem (9.2): A compact 3-manifold whose boundary is a 2-sphere is a 3-cell if and only if it can be imbedded in 3-space. Theorem (9.3): A 2-sphere is tamely imbedded in 3-space if the induced imbedding of each hemisphere is tame. (The analogous statement about a 1-sphere in 3-space is known to be false.) *R. H. Fox* (Princeton, N. J.).

Thom, René. Quelques propriétés globales des variétés différentiables. Comment. Math. Helv. **28**, 17–86 (1954).

This paper gives a complete exposition with proofs of some results previously announced by the author in four notes [C. R. Acad. Sci. Paris **236**, 453–454, 573–575, 1128–1130, 1733–1735 (1953); these Rev. **14**, 1005, 1112].
W. S. Massey (Princeton, N. J.).

Mostert, Paul S. Fibre spaces with totally disconnected fibres. Duke Math. J. **21**, 67–74 (1954).

The author defines a fiberable space as a system $[X, B, \pi]$ consisting of two topological spaces X and B and an open continuous map π of X onto B. By adding appropriate further conditions a fiberable space becomes (according to the added conditions) a fibre bundle, a fibre space in the sense of Hu, or an E-F bundle [see Steenrod, The topology of fibre bundles, Princeton, 1951; these Rev. **12**, 522]. Various relations between these concepts are studied in the case of totally disconnected fibres and applications are made to topological groups. *D. Montgomery.*

Fadell, Edward R. Identifications in singular homology theory. Pacific J. Math. **3**, 529–549 (1953).

This paper extends several known theorems on identifications in singular homology theory [T. Radó, same J. **1**, 265–290 (1951); these Rev. **13**, 373; P. V. Reichelderfer, ibid. **2**, 73–97 (1952); these Rev. **13**, 765; see these papers for definitions]. The principal theorem: Let β_p be the barycentric map of the Radó chain complex $\{C_p^R\}$ of a space into itself and let σ_p be the natural map of C_p^R into the Eilenberg chain complex. Then the kernels $\{K_p\}$ of the homomorphisms $\sigma_p\circ\beta_p^R$ are unessential identifiers, in the sense that the natural maps of the homology groups of $\{C_p^R\}$ into those of $\{C_p^R/K_p\}$ are isomorphisms onto. It is also shown that the kernels of the barycentric map of the Eilenberg chain complex into itself are unessential identifiers.
J. L. Kelley (Berkeley, Calif.).

Stein, S. K. Homology of the two-fold symmetric product. Ann. of Math. (2) **59**, 570–583 (1954).

This paper studies the homology structure of the two-fold symmetric product Y of a complex K with itself. Previous results in this direction were obtained by Smith [Proc. Nat. Acad. Sci. U. S. A. **19**, 612–618 (1933)], who obtained the relations between the mod 2 homology of K and that of Y, and by Richardson [Duke Math. J. **1**, 50–69 (1935)], who showed how the Betti numbers of K and the homology of K mod p^n (p an odd prime) determine the Betti numbers of Y and the homology of Y mod p^n. The present work completes the problem of determining the homology of Y from that of K by showing how the integral homology groups of Y are obtained from those of K.

The method uses the special homology theory previously used by Smith and Richardson. If $\rho\colon K\to K$ is a chain transformation of a complex, let K^ρ = image of ρ and $K^{\bar\rho}$ = kernel of ρ. Then, corresponding to the exact sequence of complexes $0\to K^{\bar\rho}\to K\to K^\rho\to 0$, there is an exact sequence of homology groups

$$\cdots\to H_q(K)\to H_q(K^\rho)\xrightarrow{\mu} H_{q-1}(K^{\bar\rho})\to H_{q-1}(K)\to\cdots.$$

If α, β are two chain transformations of K into K such that $\alpha\beta=\beta\alpha=0$, then $K^\beta\subset K^{\bar\alpha}$ and $K^\alpha\subset K^{\bar\beta}$, so that we have the injections

$$H_q(K^\beta)\xrightarrow{j_*}H_q(K^{\bar\alpha})\quad\text{and}\quad H_q(K^\alpha)\xrightarrow{j_*}H_q(K^{\bar\beta})$$

and the diagram

$$\begin{array}{lll}
 & H_q(K^\alpha)\xrightarrow{\mu} & H_{q-1}(K^{\bar\alpha}) \\
 & \quad\nearrow^{j_*} & \\
 & H_{q-1}(K^\beta)\xrightarrow{\mu} & H_{q-2}(K^{\bar\beta}) \\
 & \quad\nearrow^{j_*} & \\
 & H_{q-2}(K^\alpha)\to & \\
 & \quad\vdots &
\end{array}$$

If $a\in H_q(K^\alpha)$, it defines a cascade of elements as follows. If there is $b\in H_{q-1}(K^\beta)$ such that $j_*b=\mu(a)$, we see if $\mu(b)$ is in the image of j_*. Continuing in this way we get a sequence of elements connected by the homomorphisms μ and j_*, and this sequence stops when an element is obtained which is not in the image of j_*.

Let $X=K\times K$ subdivided barycentrically and let $T\colon X\to X$ be the simplicial map defined by the vertex map $T(x_1\times x_2)=x_2\times x_1$. Let $\sigma=I+T$, $\tau=I-T$ where I denotes the identity map of X. Then Y is the orbit complex of X under T, so that the homology groups of Y are related to the special homology groups $H_q(X^\tau)$. The latter are determined by using the Künneth formulas and the cascades.

Explicit relations between the homology of K and Y are listed at the end of the paper. For example, if $p(t)$ denotes the Poincaré polynomial of K and $Q(t)$ that of Y then

$$Q(t)=\tfrac{1}{2}(p(t))^2+\tfrac{1}{2}p(t^2),$$

and this relation also holds for the Poincaré polynomials defined using homology mod p^n where p is an odd prime. A more complicated formula relates the Poincaré polynomials mod 2.

The work is solely concerned with homology and no mention is made of relations which might exist between the multiplicative structure of K and of Y. *E. H. Spanier.*

Deheuvels, René. Cohomologie d'Alexander-Čech à coefficients dans un faisceau sur un espace topologique quelconque. Applications. C. R. Acad. Sci. Paris **238**, 1089–1091 (1954).

Deheuvels, René. Filtration d'Alexander-Čech de la cohomologie singulière. Répartition des points critiques d'une fonction numérique. C. R. Acad. Sci. Paris **238**, 1186–1188 (1954).

Deheuvels, René. Expression des différentielles δ_r de la suite spectrale d'une application continue. C. R. Acad. Sci. Paris **238**, 1286–1288 (1954).

The first of these notes is concerned with a general definition of the Alexander-Čech type cohomology of an arbitrary space with coefficients in a sheaf (faisceau). This is then used to give a definition of the spectral sequence of a continuous map from one space to another.

The second and third notes rely on the concepts introduced in the first. The second note defines a spectral sequence whose term E_2 is the Alexander-Čech cohomology of X with coefficients in the sheaf of singular cohomology of X and whose limit is the singular cohomology of X. This is applied to the study of the homology properties of the critical set of a real-valued function defined on X.

The third note shows how the differential operators of the spectral sequence of a map $X \to Y$ can be determined in terms of natural homomorphisms connecting the cohomology of Y with coefficients in different sheaves.

E. H. Spanier (Chicago, Ill.).

Gordon, W. L. On the coefficient group in cohomology. Duke Math. J. **21**, 139–153 (1954).

This paper investigates the effect of homomorphisms of the coefficient group on the Alexander-Wallace cohomology groups of compact Hausdorff spaces. To any homomorphism $h\colon G \to H$ of the coefficient groups there is associated a homomorphism $h^*\colon H^p(X, A; G) \to H^p(X, A; H)$ of the corresponding cohomology groups. This associated homomorphism has the obvious naturality properties, and given an exact sequence $0 \to K \to G \to H \to 0$, the above homomorphisms fit into an exact sequence

$$\cdots \to H^p(X, A; K) \to H^p(X, A; G) \to H^p(X, A; H) \to H^{p+1}(X, A; K) \to \cdots.$$

One of the main results asserts that if G is the direct limit of the groups G_λ then $H^p(X, A; G)$ is the direct limit of the groups $H^p(X, A; G_\lambda)$. Some applications of these concepts are presented.

E. H. Spanier (Chicago, Ill.).

Miyazaki, Hiroshi. On generalizations of Hopf's classification theorems. Tôhoku Math. J. (2) **5**, 284–289 (1954).

Using the notion of paracompactness and the notion of a space dominated by a CW-complex, the author gives in Theorem I the expected generalization of the Hopf Classification Theorem [H. Whitney, Duke Math. J. **3**, 51–55 (1937); S. Eilenberg, Ann. of Math. (2) **41**, 231–251 (1940); these Rev. **1**, 222]. In theorem II, he gives another generalization. Here, it would seem that by an error in notation the author has used ordinary Čech cohomology instead of Čech cohomology based on finite coverings.

J. C. Moore.

Miyazaki, Hiroshi. The cohomotopy and uniform cohomotopy groups. Tôhoku Math. J. (2) **5**, 83–103 (1953).

In this paper the author shows that cohomotopy groups may be defined for paracompact spaces instead of merely for compact spaces subject to the usual dimensional restrictions [E. Spanier, Ann. of Math. (2) **50**, 203–245 (1949); these Rev. **10**, 559]. He also shows that another sort of cohomotopy groups, uniform cohomotopy groups, may be defined on the same class of spaces, where uniform homotopy replaces ordinary homotopy in the definition of equivalence between two mappings. Further, he shows that for a paracompact pair there is both a cohomotopy sequence and a uniform cohomotopy sequence, and that both are exact.

J. C. Moore (Paris).

Aoki, Kiyoshi. On some invariants of mappings. Tôhoku Math. J. (2) **5**, 220–237 (1954).

The author asserts that he has obtained an integer-valued function W_r of homotopy classes of maps $S^m \to S^n$, $m = (r+1)n - r$, n even, which generalizes the Hopf invariant, and that this function leads to precise analogues of the Freudenthal suspension theorems. However, his two main theorems lead jointly to contradiction. Theorem 4.1 asserts that the elements of $\pi_m(S^n)$ with $W_r = 0$ constitute $E\pi_{m-1}(S^{n-1})$, and Theorem 4.2 asserts that E is (1-1) on the set of elements of $\pi_m(S^n)$ with $W_r = 0$; but if we put $n = 4$, $m = 10$, then $E\pi_q(S^3) = Z_3$ whereas $E^2\pi_q(S^3) = 0$.

It appears to the reviewer that, in fact, $W_r = 0$ for $r > 1$; for W_r seems from its definition to be a homomorphism and it is known from Serre's result that $\pi_m(S^n)$ is finite if $r > 1$. However, the process of obtaining the particular intersection number whereby W_r is defined depends on ideas introduced in a previous paper [same J. (2) **4**, 178–186 (1952); these Rev. **14**, 673], and these are not of a kind to lend themselves readily to comprehension.

P. J. Hilton (Cambridge).

Uehara, Hiroshi. On a homotopy classification of mappings of a four-dimensional polyhedron into a simply-connected space with vanishing 3-rd homotopy group. J. London Math. Soc. **29**, 292–309 (1954).

In a previous paper [Nagoya Math. J. **4**, 43–50 (1952); these Rev. **13**, 859], N. Shimada and the present author have given a homotopy classification theorem for maps of an $(n+2)$-dimensional polyhedron into an arcwise connected space Y whose homotopy groups $\pi_i(Y)$ vanish for $i < n$ and for $i = n+1$ (and whose n-dimensional homotopy group is finitely generated) in the case where $n > 2$. In the present paper, the author studies this problem for the case $n = 2$. The methods used are similar to those of the previous paper. For the purposes of his main theorem, he introduces three new cohomology operations.

W. S. Massey.

Sugawara, Masahiro. Some remarks on homotopy groups of rotation groups. Math. J. Okayama Univ. **3**, 129–133 (1954).

In this note the author continues his investigations of the homotopy groups of rotation groups [see Sugawara, same J. **3**, 11–21 (1953); these Rev. **15**, 457; the notations are preserved in the later note]. The groups $\pi_9(R_n)$ are completely calculated, with explicit generators, and the groups $\pi_{10}(R_n)$, $\pi_{11}(R_n)$ are determined modulo certain ambiguities.

Lemma 1 may be improved to $E^{n+1}p_*\Delta\alpha = 0$, n odd, $= 2E^n\alpha$, n even, where $\alpha \in \pi_{r+1}(S^n)$, since $H_0[\alpha, \iota_n] = 0$, n odd, $= 2E^n\alpha$, n even; of course, this lemma is significant only if α non-$\in E\pi_r(S^{n-1})$. Proposition 1 is contained in a theorem due to B. Eckmann [Comment. Math. Helv. **15**, 318–339 (1943), p. 326; these Rev. **5**, 104]. In Proposition 2, "$2\epsilon_7 = \zeta_7$" should read "$2\zeta_7 = \epsilon_7$". The other misprints should cause no difficulty.

P. J. Hilton (Cambridge, England).

James, I. M. On the iterated suspension. Quart. J. Math., Oxford Ser. (2) **5**, 1–10 (1954).

Let R_{m+n} denote the group of proper rotations of Euclidean space of $m+n$ dimensions, and $R_n \subset R_{m+n}$ the subgroup of rotations in the plane of the first n axes. The author defines homomorphisms

$$H_m\colon \pi_{r+n+m}(S^{n+m}) \to \pi_r(R_{n+m}, R_n) \quad \text{for} \quad r \leq 2n-2$$

and

$$P_m\colon \pi_r(R_{n+m}, R_n) \to \pi_{r+n-1}(S^n) \quad \text{for all } r.$$

Let $E^m\colon \pi_q(S^n) \to \pi_{q+m}(S^{n+m})$ denote the resultant of m successive Freudenthal suspensions. The main theorem asserts that homomorphisms $E^m, H_m, P_m, E^m, \cdots$ (when taken in this order) constitute an exact sequence. When $m = 1$, one obtains a variant of an exact sequence given by G. W.

Whitehead [Ann. of Math. (2) 57, 209–228 (1953); these Rev. 14, 1110]. These exact sequences are applied to obtain various relations between the homotopy groups of spheres, rotation groups, Stiefel manifolds, etc.

W. S. Massey (Princeton, N. J.).

James, I. M., and Whitehead, J. H. C. Note on fibre spaces. Proc. London Math. Soc. (3) 4, 129–137 (1954).

This note contains various theorems and lemmas which the authors use in the paper reviewed below. Most of them are concerned with the homotopy type of spaces. As samples of the results obtained, the following may be quoted. (1) Let $f: X \rightarrow Y$ be a fibre mapping (in the sense that the covering homotopy property holds), and let A be a fibre. Assume that X, A, and Y are arcwise connected and satisfy appropriate local smoothness conditions. If A is a retract of X, then X has the same homotopy type as the product space, $Y \times A$. (2) Let X and X' be connected fibre bundles with base space an n-sphere and fibres A and A' respectively. Assume that A and A' are finite polyhedra with finite fundamental groups, that X and X' are of the same homotopy type, and A and A' are of the same homotopy type. Then if either of the bundles X or X' admits a cross-section, so does the other.

W. S. Massey (Princeton, N. J.).

James, I. M., and Whitehead, J. H. C. The homotopy theory of sphere bundles over spheres. I. Proc. London Math. Soc. (3) 4, 196–218 (1954).

The authors consider fibre bundles in which the base sapce is an n-sphere, S^n, the fibre is a q-sphere, S^q, and the structural group is the rotation group of S^q, denoted by R_{q+1}. It is a classical result of Feldbau that the equivalence classes of such bundles are in 1-1 correspondence with the elements of the homotopy group $\pi_{n-1}(R_{q+1})$. Examples are known of pairs of bundles which are not equivalent from the point of view of the general theory of bundles, but the homotopy type of the bundle spaces can not be distinguished by means of any of the known algebraic invariants [see N. E. Steenrod, The topology of fibre bundles, Princeton, 1951, §26; these Rev. 12, 522]. In this paper, the authors consider the problem of distinguishing the homotopy type of such bundles in the case of bundles admitting a cross-section. For every such bundle B they introduce an invariant $\lambda(B)$ which is an element of a factor group of $\pi_{n+q-1}(S^q)$. This invariant may be interpreted as the $(q+1)$th obstruction to the retraction of B onto one of its fibres; if B is a product bundle, then $\lambda(B)=0$. They are able to give necessary and sufficient conditions for two such bundles to have bundle spaces of the same homotopy type in terms of this invariant.

The authors apply these general results to various specific problems. Among the specific results thus obtained, the following may be cited: (1) Let W^{2n-1} denote the Stiefel manifold R_{n+1}/R_{n-1}, which may be identified with the space of unit tangent vectors to S^n. Then W^{2n-1} is of the same homotopy type as $S^n \times S^{n-1}$ if and only if $\pi_{2n+1}(S^{n+1})$ contains an element of Hopf invariant one. (2) Let $W_{n,r}$ denote the complex Stiefel manifold of ordered r-tuples of orthonormal vectors in the space of n complex variables. Then $W_{k+1,2}$ has the same homotopy type as $S^{2k+1} \times S^{2k-1}$ if and only if $\pi_{4k+3}(S^{2k+2})$ contains an element of Hopf invariant one. An analogous result also holds for quaternionic Stiefel manifolds. (3) The authors explicitly determine all homotopy types of q-sphere bundles over an n-sphere which admit a cross-section for $1 \leqq n \leqq 6$ and $q \geqq 1$. An interesting case is that of bundles over S^4 having S^3 as a fibre. It is known that there are a countable infinitude of equivalence classes of such bundles which admit a cross-section. Steenrod [loc. cit, §26.6] denotes these bundles by $B_{k,0}$, where k ranges over all integers. The present authors show that $B_{k,0}$ and $B_{m,0}$ are of the same homotopy type if and only if $k \equiv \pm m \bmod 12$.

W. S. Massey (Princeton, N. J.).

Andreoli, Giulio. Preliminari topologici su gli alberi. Giorn. Mat. Battaglini (5) 2(82), 237–266 (1 plate) (1954).

The author defines a tree as a finite set of points and segments or arcs, such that each segment joins two points, each point belongs to at least one segment, and there is no closed polygon formed by the segments and points. [This definition does not imply connectedness, which the text appears to assume.] The following is concluded. The theory of trees in space can be reduced to the planar case, though different plane trees can be derived from the same spatial one. The theory of plane trees in turn reduces to that of plane trees with an origin, and thence to the study of change of origin in a fixed tree and suppression or adjunction of nodes and branches. A tree can be characterized (uniquely to within equivalence) by a matrix of a certain type describing the relationship of sides of polygons bordering the tree, or equivalently by the sequence of sums of columns of this matrix. It is likewise characterized by a sequence of couples of numbers, the numbers in a couple describing the number of steps in the paths to two different branch tips from the point where the paths from the origin separate. The author studies the relationship of these characteristics of a tree to the characteristics of related trees, such as trees whose union it is. The work concludes with a detailed example. It is stated that this paper was finished 10 years ago and is independent of other work of the period.

P. M. Whitman (Silver Spring, Md.).

GEOMETRY

Castrucci, Benedito. On the method of Denise-Gastão Gomes. Bol. Soc. Mat. São Paulo 4 (1949), 19–29 (1951). (Portuguese)

Fischer, Helmut Joachim. Geometrische Netze und Konfigurationen und ihre Beziehungen zur Vektorrechnung und Zahlentheorie. II. Collectanea Math. 6, 3–89 (1953).

The author applies the methods of Part I [Collectanea Math. 4, no. 2, 57–119 (1951); these Rev. 14, 785] to the investigation of "nets" K_{III}, K_{IV}, K_V, K_{VI}, derived from three, four, five or six points on a conic. In K_{III}, for instance, the three points A_1 and the corresponding three tangents a_1 yield 3 joins $a_2=A_1A_1$ and 3 intersections $A_2=a_1\cdot a_1$ ("of degree 1"), 3 joins $a_3=A_1A_2$ and 3 intersections $A_3=a_1\cdot a_2$ ("of degree 2"), 3+1 joins $a_4=A_2A_3$, $a_5=A_3A_3$ and 3+1 intersections $A_4=a_2\cdot a_3$, $A_5=a_3\cdot a_3$ ("of degree 3"), and so on. The details are worked out up to degree 5 for K_{III}, up to degree 4 for K_{IV}, up to degree 3 for K_V and K_{VI}. In the last case the results are related to the Pascal configuration with its Steiner points, Kirkman points, Steiner-Plücker lines, Cayley-Salmon lines, etc.

H. S. M. Coxeter.

Ladopoulos, Panaïotis D. Une extension d'un théorème de Clifford. C. R. Acad. Sci. Paris 238, 2050–2051 (1954).

Let (C_1), (C_2), $\cdots$ be members of a range of conics in the complex Euclidean plane. Let (C_{ij}) be the conic through a given point M and the four intersections of (C_i) and (C_j). Four conics (C_1), $\cdots$, (C_4), taken in pairs, determine six such conics (C_{12}), $\cdots$, (C_{34}), having three common points which are said to be "associated" with M. Any three of the four conics determine a circular quartic passing through the twelve intersections of the three conics. The four conics thus determine four quartics. These quartics have ten common points, six of which are "associated" with the circular points at infinity. The remaining four points are said to be associated with the set of four conics. Five conics of the same range, taken in sets of four, determine five such tetrads of points, all lying on a circular quartic. Six conics of the same range, taken in sets of five, determine six such circular quartics having four common points; seven conics determine seven such tetrads lying on a circular quartic; and so on.

H. S. M. Coxeter (Toronto, Ont.).

Mandzyuk, A. I. On some many-valued correspondences in projective geometry. Ukrain. Mat. Žurnal 5, 439–452 (1953). (Russian)

This is a critical survey of the dissertation of K. A. Andreev [Mat. Sbornik **9**, 361–434 (1879)], the dissertation of A. K. Vlasov [Učenye Zapiski Imp. Moskov. Univ. Otd. Fiz.-Mat. **25** (1911)] and paper of A. A. Glagolev [C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 291–292 (1946); these Rev. **8**, 483]. It is pointed out that an incomplete proof of Andreev on the construction of further corresponding pairs in a 2-2 correspondence between two line pencils from 9 given pairs was later (independently) completed by R. Sturm. The only new contributions of the author are: (1) an outline of a method of filling similar gaps in the construction of further corresponding pairs in 3-3 correspondences from 15 given pairs; (2) a method of making Vlasov's results more easily accessible by using higher-dimensional spaces.

H. Busemann (Copenhagen).

Lenz, Hanfried. Kleiner Desarguesscher Satz und Dualität in projektiven Ebenen. Jber. Deutsch. Math. Verein. 57, Abt. 1, 20–31 (1954).

There is a discussion of duality in planes in which the minor theorem of Desargues is valid in various local applications. It is shown in particular that a plane coordinated by a near-field which is not a division ring is never self-dual. This is shown by proving that both distributive laws must hold.

Marshall Hall, Jr. (Columbus, Ohio).

Klingenberg, Wilhelm. Eine Begründung der hyperbolischen Geometrie. Math. Ann. 127, 340–356 (1954).

Following Hjelmslev [Math. Ann. **64**, 449–474 (1907)], the author identifies the points and lines of a plane with the corresponding congruent transformations: point-reflections and line-reflections. In the case of hyperbolic geometry, these can be regarded as projectivities on the absolute conic, and hence as linear fractional transformations (of positive or negative determinant, respectively). The group of linear fractional transformations (over a field of characteristic $\neq 2$) is generated by involutory elements called line-reflections or lines. Two distinct lines, g and h, are said to be orthogonal if $gh=hg$, and then their product is called a point-reflection or point: $gh=hg=P$. The condition for incidence is $Pg=gP$. Three lines, g, h, j, are said to belong to a pencil of $ghj=jhg$. Two lines are said to be parallel if they have neither a common point nor a common perpendicular. The author uses five of the axioms of absolute geometry of F. Bachmann [ibid. **123**, 341–344 (1951); these Rev. **13**, 767] and adds three more to make the geometry hyperbolic: (A6) there exist three lines, g, h, j, such that h is perpendicular to g and parallel to j. (A7) for any two pairs of parallel lines, there exists a line that belongs to a pencil with each pair. (A8) if a line is parallel to each of three lines, it belongs to a pencil with at least one pair of them. These axioms are shown to be equivalent to Bachmann's characterization of the linear fractional group [ibid. **126**, 79–92 (1953); these Rev. **15**, 197].

H. S. M. Coxeter (Toronto, Ont.).

Algebraic Geometry

***Hodge, W. V. D., and Pedoe, D. Methods of algebraic geometry. Vol. III. Book V: Birational geometry.** Cambridge, at the University Press, 1954. x+336 pp. $7.50.

In this third volume of their treatise on algebraic geometry the authors propose—and we quote from the preface—"to provide an account of the modern algebraic methods available for the investigation of the birational geometry of algebraic varieties". Thus, while in the preceding two volumes of this treatise [1947, 1952; these Rev. **10**, 396; **13**, 972] the algebraic methods used belonged primarily to linear algebra, elimination theory and field theory, in the present volume the authors are chiefly concerned with some relatively advanced questions in birational geometry the solution of which requires a generous dose of ideal theory and valuation theory. The treatment of these questions is taken largely from the work of the reviewer, and this is the first time that such material, scattered in many original memoirs, and of recent vintage, has been gathered together and presented in book form. The result is a very readable and self-contained account of a modern phase of algebraic geometry. Following their policy inaugurated in the preceding volume, the authors restrict their exposition to ground fields of characteristic zero. Since they are intent on giving an account of methods as well as of results, the authors often give more than one proof of one and the same theorem, using different points of view and thus throwing additional light on the subject.

The volume consists of the following 4 chapters (chapters XV–XVIII of the treatise): ch. XV: Ideal theory of commutative rings; ch. XVI: The arithmetic theory of varieties; ch. XVII: Valuation theory; ch. XVIII: Birational transformations. Chapter XV develops the main facts about Noetherian rings, quotient rings and integrally closed Noetherian domains. This standard material from ideal theory is applied in the next chapter to a study of some basic arithmetic properties of algebraic varieties. After some preliminaries concerning the relationship between ideals, varieties (both affine and projective) and the coördinate rings of a variety, the authors give a detailed account of the reviewer's theory of simple points and simple subvarieties (for characteristic zero only), including such topics as the intrinsic characterization of simple points by means of their quotient rings, properties of local uniformizing parameters, formal power series expansions at simple

points, etc. The proofs are often quite different from those given originally by the reviewer, since the authors use as starting point their definition of simple points, given in the second volume and based on intersection multiplicities, and, furthermore, use a good deal the properties of the associated form of a variety. The authors then introduce and study the concept of a normal variety (both in the affine and the projective space), prove the existence of a derived normal variety of any given variety and give a characterization of a projectively normal variety ("arithmetically normal", in the terminology of the reviewer) by means of the completeness of the linear systems cut out on the variety by hypersurfaces of all orders. In chapter XVII, Krull's general valuation theory is developed, with particular emphasis on valuations of finite rank and dimension-theoretic properties of valuations of algebraic function fields. The chapter also includes a section on the concept of the center of a valuation, due to the reviewer. The culminating point of the book is reached in the last chapter where some birational geometry proper is treated. The chapter contains 9 sections and is devoted exclusively to the exposition of some results due to the reviewer. In section 1 the authors develop the arithmetic theory of birational transformations based on the theory of valuations. In section 2 this general theory of birational transformations is applied to the theoretically important case of normal varieties. (However, the "Main Theorem" of the reviewer is given without proof.) Section 3, preliminary to the treatment of the local uniformization problem, introduces and studies monoidal transformations. The next 4 sections (4–7) are given entirely to the proof of the difficult theorem on local uniformization. The authors have made a considerable effort to render this proof as accessible as possible. They also include in their exposition a derivation of Perron's algorithm of simultaneous approximations of finitely many irrational numbers. In section 8 the existence of finite resolving systems is proved (the original bicompactness considerations of the reviewer are replaced by equivalent algebraic considerations). In the last section the authors give the reviewer's "simplified proof" of the theorem of reduction of singularities of algebraic surfaces. [Some of the proofs in the section on simple points (ch. XVI) seem to the reviewer to be unnecessarily long. For instance, the proof of theorem II, on p. 143, ends on p. 145, but already in the first few lines of p. 144 it is shown that the maximal ideal of the quotient ring $\mathfrak{J}_P$ is generated by the single element ω, and hence the theorem now follows directly from theorem IX, p. 139. Similarly, in regard to the proof of the converse of theorem VII, p. 132, it may be observed that if $\mathfrak{P}$ denotes the prime ideal of V in $K[X_1, X_2, \cdots, X_n]$ and if, say, $\zeta_i=\varphi_i(\xi)$, $i=1, 2, \cdots, d$ (where it is permissible to assume that the φ_i are polynomials), then from the fact (established on the first few lines of p. 133) that $\zeta_1, \zeta_2, \cdots, \zeta_d$ are uniformizing parameters at $x^{(k)}$ follows that $\varphi_1(X), \varphi_2(X), \cdots, \varphi_d(X)$, together with a basis of $\mathfrak{P}$, form a set which contains a system of uniformizing parameters of $x^{(k)}$ in the affine space A_n. It follows that $\mathfrak{P}$ contains $n-d$ polynomials $f_i(X)$ such that the hypersurfaces $f_i(X)=0$ have independent tangent hyperplanes at the point $x^{(k)}$. It now seems to the reviewer that this, together with the fact that V is d-dimensional, must imply, according to the definition of simple points as given in the second volume, that $x^{(k)}$ is a simple point of the variety V over K^*. If that is so, then the proof of the relation $K'\subset\mathfrak{J}_p$, which takes up pages 133–136, can be omitted.]

O. Zariski (Cambridge, Mass.).

Bergman, Gösta. On the exceptional points of cubic curves. Ark. Mat. 2, 489–535 (1954).

According to T. Nagell [Skr. Norske Vid. Akad. Oslo. I. **1935**, no. 1 (1936)] a point (x, y) of the elliptic cubic curve

$$(1) \qquad y^2=x^3-Ax-B \quad (4A^3-27B^2\neq 0)$$

is called an exceptional point of order n, if—in connection with the classical parametric representation of the curve (1) by means of elliptic functions $x=\wp(u)$, $y=\frac{1}{2}\wp'(u)$, with invariants $4A$, $4B$ and a pair of primitive periods ω, ω'—the value of the parameter u which corresponds to (x, y) (defined by this point mod ω, ω') is commensurable with a period, n being the smallest natural number that makes nu a period. If A and B belong to a field Ω, the values of u belonging to those exceptional points which have their coordinates in Ω constitute an Abelian group, called the exceptional group in Ω on the curve (1) [F. Châtelet, C. R. Acad. Sci. Paris **210**, 90–92 (1940); these Rev. **1**, 166]. On account of a result of A. Weil [Acta Math. **52**, 281–315 (1929)], this group is finite if Ω is an algebraic field; then the group contains at most two independent elements having as order any assigned prime [G. Billing, Nova Acta Soc. Sci. Upsaliensis (4) **11**, no. 1 (1938)], and so there is only a finite number of a priori possible structures, indicated here by a simple notation, for an exceptional group of a given order.

In the present paper, parametric expressions for A and B are obtained in correspondence with groups of several types, and precisely of the types (4, 2), (8), (2, 5), (2, 2, 3), (4, 3), (4, 4), (8, 2), (2, 3, 3). Further, the exceptional group is determined when the curve (1) is harmonic ($B=0$) or equianharmonic ($A=0$) and Ω is an algebraic field of degree n, where $n=2$, 4 or an odd number in the harmonic case, and $n=2$, 3 or a number not divisible by 2, 3 in the equianharmonic case. The many results obtained cannot all be described, and we shall give details only on a few of them.

While C. E. Lind [Thesis, Uppsala, 1940; these Rev. **9**, 225] has shown that the groups (4, 2, 3) and (2, 2, 5) are impossible in $k(1)$, here it is proved that each of these groups is possible in an infinity of quadratic fields, even on supposing A and B to be rational. If Ω is an algebraic field of odd degree, A is an arbitrarily given number of Ω and C denotes any number of Ω, then the harmonic curve $y^2=x^3-Ax$ has the following exceptional group in Ω:

$$(2),\ \text{if } A\neq C^2, A\neq -4C^4; \quad (2, 2),\ \text{if } A=C^2; \quad (4),\ \text{if } A=-4C^4.$$

If Ω is an algebraic field of degree not divisible by 2, 3, B is an arbitrarily given number of Ω and C denotes any number of Ω, then the equianharmonic curve $y^2=x^3-B$ has the following exceptional group in Ω:

(1), if $B\neq C^2, B\neq -C^3, B\neq 432C^6$;
(2), if $B=C^2, B\neq -C^6$;
(3), if $B=-C^3$ or $B=432C^6$, but $B\neq -C^6$;
(2, 3), if $B=-C^6$.

B. Segre (Rome).

Barlotti, Adriano. Alcune osservazioni sulle quartiche piane dotate di un tacnodo simmetrico. Boll. Un. Mat. Ital. (3) **9**, 55–58 (1954).

Let Γ_4 be a plane quartic with a node A and a tacnode O, and let $M_1, M_2; N_1, N_2$ be the intersections of Γ_4 (distinct from A) with two lines through A. If both pairs M_1, N_1; M_2, N_2 are collinear with O, then the tacnode at O is harmonic. This and similar theorems on plane quartics are

given by the author, who makes some applications to a skew quartic of the 2nd kind, with a node.

V. Dalla Volta (Rome).

Fano, Gino. Les surfaces du quatrième ordre. Univ. e Politecnico Torino. Rend. Sem. Mat. 12, 301–313 (1953).

An elementary expository lecture containing brief descriptions of various special types of quartic surface, particularly the Kummer surface, wave surface, cyclide, and Steiner surface.

P. Du Val (London).

Predonzan, Arno. Sui sistemi lineari di superficie algebriche dello spazio a curva caratteristica di genere π e di grado $n\geq 3\pi+3$. Rend. Sem. Mat. Univ. Padova 23, 127–162 (1954).

The classification of complete linear systems of surfaces of S_3 whose characteristic curves are of genus $\pi(\geq 2)$ is equivalent to the classification of normal 3-dimensional varieties $V_3{}^n$, with regular surface sections and curve sections of genus π, which are the projective models of the linear systems. The classification for the cases $\pi=2$ and $\pi=3$ has been given by Morin [Ann. Mat. Pura Appl. (4) 19, 257–288 (1940); 21, 113–155 (1942); these Rev. 8, 87; 6, 102]. Here the author classifies such $V_3{}^n$ in S_r, excluding cones, for which $n\geq 3\pi+3$, and their corresponding linear systems. The detailed enumeration of the results is given in three pages of the introduction to the paper.

D. B. Scott (London).

Predonzan, Arno. Osservazioni sulle varietà algebriche a tre dimensioni a superficie irregolari. Rend. Sem. Mat. Univ. Padova 23, 245–254 (1954).

The author is concerned with conditions under which a 3-dimensional irregular algebraic variety V_3 is "pseudoplanar" (i.e., birationally equivalent to a variety of ∞^1 planes). It is shown that, if V contains an irreducible algebraic system of irregular surfaces, containing a linear system Σ_r of dimension r and such that the grade of the system is n and its characteristic curve is of genus $\pi\geq 3$, then V is necessarily pseudo-planar if either (1) Σ_r is a system of "prime sections" and $n>2\pi-2$, or (2) $r\geq 3\pi+b$. In this latter case V can be birationally transformed into a planar variety so that the characteristic curves of Σ_r are transformed into directrix curves.

D. B. Scott (London).

Burau, Werner. Grundmannigfaltigkeiten der projektiven Geometrie. III, IV. Collectanea Math. 5, 4–118 (1952).

Parts I, II of this work, dealing with the Veronese and Segre manifolds respectively, appeared in Collectanea Math. 3, no. 2, 53–163 (1950); these Rev. 13, 977.

Part III is devoted to Grassmann manifolds $G_{n;k}$, representing by points the totality of S_k's in S_n; $G_{n;k}$, is represented parametrically by equating $\binom{n+1}{k+1}$ coordinates $p_{i_0\cdots i_k}$ $(0\leq i_0\leq\cdots\leq i_k\leq n)$ to the determinants of the i_0th, $\cdots$, i_kth columns of an $(n+1)\times(k+1)$ matrix with variable elements, and is of $(k+1)(n-k)$ dimensions spanning $S_{\binom{n+1}{k+1}-1}$. An inductive construction for $G_{n;k}$ is given by taking $G_{n-1;k}$ and $G_{n-1;k-1}$ in mutually skew ambients, and joining suitably related points of the two by lines; this leads to a simple proof that $G_{n;k}$ can be defined by a set of equations all of which are quadratic. $G_{n;1}$ (the Grassmannian of lines in S_n) is studied in detail with reference to the theory of skew-symmetric matrices. In the space whose coordinates are the elements of an $(n+1)\times(n+1)$ matrix, the symmetric and skew-symmetric matrices correspond to sub-spaces mutually skew and spanning the whole space, and the conditions that the matrix is of rank 2 define in the whole space and the two sub-spaces respectively the Segre, Veronese, and Grassmann manifolds $S_{n,n}$, $V_n{}^2$, $G_{n;1}$; hence $G_{n;1}$ is the projection of $S_{n,n}$ from a $V_n{}^2$ lying on it. Further special consideration is given to the case $n=2k+1$, in which S_k is self dual in S_n, especially $G_{k;2}$.

Part IV deals with a much less familiar type of manifold, denoted by $M_{Q_{2k}}$, which is defined as the normal minimum-order image by points, one-one without exception, of the S_k's on the general quadric Q_{2k} in S_{2k+1}; the Grassmannian of these S_k's is of course such an image, except that it is not minimum order; e.g. the Grassmannians of the lines on Q_2 and the planes on Q_4 are a conic and the Veronese manifold $V_3{}^2$ (projective model of all quadrics in S_3); the corresponding minimum order models are thus a line and S_3. It is shown that an $M_{Q_{2k}}$, on which every subsystem of ∞^1 S_k's through an S_{k-2} is mapped by a line, and every subsystem of ∞^3 through an S_{k-3} by an S_3, can be constructed by joining by lines suitably related points of two $M_{Q_{2k-2}}$'s in mutually skew ambients. Thus the Grassmannian of the S_k's on Q_{2k} is simply the projective model of all quadric sections of $M_{Q_{2k}}$, though this point is not made by the author. M_{Q_6} is itself a Q_6, M_{Q_8} an M_{10} in S_{15}, and $M_{Q_{10}}$ an M_{15} in S_{31}; but the reviewer cannot find that the order of either of these is given, or any general results on the order of $M_{Q_{2k}}$, though the subvarieties of M_{Q_6}, M_{Q_8}, $M_{Q_{10}}$ are discussed in some detail. The algebraic discussion is hard to follow, and does not appear to lead to any general formulation of the equations of $M_{Q_{2k}}$.

P. Du Val.

Burau, Werner. Projektive Klassifikation der Veronese-Relationen und Kennzeichnung aller Punktmodelle für die Linienelemente, Geraden und Punkte-Paare des S_n. Ann. Mat. Pura Appl. (4) 35, 299–326 (1953).

The Veronese manifold $V_n{}^k$ is the projective model of all $k$ic hypersurfaces in S_n. As an image of the ordered point pairs in S_n the author considers not only the Segre product $\prod_{2n}^{h,k}$ of $V_n{}^h$ with $V_n{}^k$, but also the locus $P_{2n}^{h,k;p}$ defined by the parametric equations

$$x_{i_0\cdots i_n;j_0\cdots j_n;s_0\cdots s_m}=x_0{}^{i_0}\cdots x_n{}^{i_n}y_0{}^{j_0}\cdots y_n{}^{j_n}\pi_0{}^{s_0}\cdots\pi_m{}^{s_m}$$
$$\left(i_0+\cdots+i_n=h;\ j_0+\cdots+j_n=k;\ m=\binom{n+1}{2}-1,\ s_0+\cdots+s_m=p\right),$$

where $(x_0,\cdots,x_n)$, $(y_0,\cdots,y_n)$ are the coordinates of the two points, and $(\pi_0,\cdots,\pi_m)$ the Plücker coordinates of the line joining them; on this, unlike $\prod_{2n}^{h,k}$, the image of a coincident point-pair depends on the limiting position of the joining line. Thus, either by putting $k=0$, or by identifying the two points and writing h instead of $h+k$, we obtain a locus $J_{n;1,0}^{h,k}$ as image of the line elements (i.e. figures consisting of a line and a point in it) of S_n. Many results are obtained on how these various loci occur as subvarieties on each other, or as projections of each other from the ambients of specified subvarieties. It is also shown that these and their projections are, under certain reasonable restrictions, the most general images of the aggregates in question.

The last section of the paper is devoted to the study of the system of quadrics of which $V_n{}^k$ is the intersection, mapped on the points of a "relation-space" R. First, for the normal rational curve $V_1{}^k$ of even order, we single out the "fundamental quadric", with respect to which each point of the curve is the pole of its own osculating hyperplane. Then

"fundamental cones" are defined as those projecting, from osculating spaces of odd (even) dimensions of the curve of even (odd) order the fundamental quadric of the projection of the curve. For the general V_n^k the definitions are similar but too complicated to quote here in detail, depending first on the projection of V_n^k into a rational curve from a suitably chosen subspace of its ambient.

The images in R of the families of fundamental cones of various ranks are loci

$$J_{n;1,0}^{2,2k-4},\ J_{n;1,0}^{4,2k-8},\ \cdots,\ J_{n;1,0}^{k,0} \quad\text{or}\quad J_{n;1,0}^{k-1,2}$$

according as k is even or odd. The ambients of these loci are skew to each other and span the whole of R, i.e. the fundamental cones are a base for all the quadrics through V_n^k; and a general collineation in S_r induces in R a collineation of which the ambients of these $J_{n;1,0}^{2i,2k-4i}$'s are the invariant spaces. *P. Du Val* (London).

Abellanas, Pedro. Primals of an algebraic variety. Revista Mat. Hisp.-Amer. (4) **13**, 255–282 (1953). (Spanish)

V_r is an algebraic variety over a ground field k in a projective space S_n and $(\xi_0, \xi_1, \cdots, \xi_n)$ is a generic point of V_r, and $P=k[\xi_0, \xi_1, \cdots, \xi_n]$. A 'primal' of V_r is, in this paper, an irreducible subvariety of V_r which is the complete intersection of V_r with a primal of S_n. The main object of this paper is to give a constructive proof of the theorem that given any finite number of subvarieties of V_r whose dimensions do not exceed $r-2$, there is a "primal" of V_r containing them all.

This idea is used to discuss the simple subvarieties of V_r. A "canonical over-variety" of a subvariety V_s of V_r is a variety V_{s+1} whose defining ideal in the ring P is a prime ideal of the form $P(\Phi_1, \cdots, \Phi_{r-s-1})$, where Φ_i $(i=1, \cdots, r-s-1)$ is a form in $(\xi_0, \cdots, \xi_n)$. Among other results it is shown that V_s is simple on V_r if and only if it is simple on some canonical over-variety.

If V_r' is a birational transform of V with ring P' such that $P\subset P'$, the transformation from V to V' is called an "anti-projection". Among other properties of these transformations it is shown that the singularities of a curve can be resolved by a transformation of this type. *D. B. Scott* (London).

Abellanas, Pedro. Primals of an algebraic variety. Revista Mat. Hisp.-Amer. (4) **13**, 283–310 (1953).

This is intended, with very minor alterations, as an English translation of the preceding paper (which should be regarded as the authentic text). *D. B. Scott.*

Abellanas, Pedro. Some corrections. Revista Mat. Hisp.-Amer. (4) **13**, 310–311 (1953). (Spanish)

The author deals with misprints in, and criticism of, two earlier papers [same Revista (4) **11**, 159–179 (1951); **12**, 79–101 (1952); these Rev. **13**, 979; **14**, 314]. *D. B. Scott* (London).

Weil, André. Remarques sur un mémoire d'Hermite Arch. Math. **5**, 197–202 (1954).

Let $f(x)$ be a polynomial in x of the 4th degree, with coefficients in a field k of characteristic different from 2 and 3. The quadratic and cubic invariants i, j and the biquadratic and sextic covariants $g(x)$, $h(x)$ of $f(x)$ are then connected by the Hermite-Cayley syzygy [Hermite, J. Reine Angew. Math. **52**, 1–17 (1856) = Oeuvres, t. I, Gauthier-Villars, Paris, 1905, pp. 350–371]:

$$4g^3-if^2g-jf^3=h^2.$$

Hence the formulae $\xi=z^{-2}g(x)$, $\zeta=z^{-3}h(x)$ define a rational correspondence between the curve (1): $z^2=f(x)$ and the curve (2): $\zeta^2=4\xi^3-i\xi-j$.

This result is here completed by showing that, if $f(x)$ has a non-zero discriminant, the elliptic curve (2) is the Jacobian of the elliptic curve (1), i.e., there is a birational correspondence, defined in k, between the points of the curve (2) and the divisor classes of order zero (or 2) of the curve (1); moreover, (1) and (2) are birationally equivalent in an algebraic extension of k. The paper terminates by stating some interesting questions concerning certain extensions, and the classification of the elliptic curves on a given field whose Jacobian is an assigned elliptic curve. *B. Segre* (Rome).

Néron, André, et Samuel, Pierre. La variété de Picard d'une variété normale. Ann. Inst. Fourier Grenoble **4** (1952), 1–30 (1954).

The present paper shows the existence of a Picard variety P for every normal projective algebraic variety V. The result is well known in the "classical" case when the universal domain is the field of complex numbers, and has been recently obtained in full generality by Chow [Amer. J. Math. **76**, 453–476 (1954); these Rev. **15**, 823] in the case of Jacobian varieties, i.e. when V is a curve. The general result is proved here by using the last special case and following Picard's old idea of fibering V conveniently by means of algebraic curves, and then studying certain subgroups of the group of the Jacobian variety of the generic fibre. Thus it is shown that, if V is defined (and normal) on a given field k and we denote by $\mathbf{G}_a$, $\mathbf{G}_l$, respectively, the groups of the divisors of V which are algebraically and linearly equivalent to zero, then: (a) there is a finite algebraic extension k' of k and an Abelian projective variety A defined in k, such that in k' there is a rational isomorphism of $\mathbf{G}_a/\mathbf{G}_l$ on A; (b) there is a finite algebraic extension k'' of k and an Abelian projective variety P defined in k, such that in k'' there is a birational isomorphism i of $\mathbf{G}_a/\mathbf{G}_l$ on P; (c) there is a Poincaré family, i.e. an algebraic system (D) of divisors on V each of which is algebraically equivalent to zero, so that (D) is birationally related to P in k'', in such a way that the point of P which corresponds to a divisor D of the family is the image by i of the linear system $|D|$ determined by D. Hence the Picard variety P is determined by V but for an isomorphism. *B. Segre* (Rome).

Boughon, Pierre. Enveloppes d'une famille à un paramètre de variétés de dimension $n-1$ dans un espace de dimension n. C. R. Acad. Sci. Paris **238**, 641–644 (1954).

Let $k\subseteq K\subset\Delta$ be field of characteristic $p(\geqq 0)$ such that K is perfect and Δ is a universal domain over K. In the polynomial ring $k[T; X_1, \cdots, X_n]=k[T; X]$, let $f(T; X)$ be irreducible, f non-ε $k[T; X^p]$, f non-ε $k[T^p; X]$. Consider the family F of divisors defined by $f(a; X)=0$, where $a \varepsilon \Delta$; let V be a generic divisor of F (over K). An envelope of F is any rational cycle over K every irreducible component E of which satisfies the following: If $E\cdot V=\sum a_iY_i$, then, for at least one i, E and V are tangent at a general point of Y_i over $K(t)$. If $p=0$ the envelope satisfies the equation obtained by eliminating T between f and $\partial f/\partial T$. If $p>0$, one must allow in addition the equation obtained by eliminating T between f and polynomials in $k[T; X^p]$. *I. S. Cohen* (Cambridge, Mass.).

Differential Geometry

Crommelin, C. A., and van der Woude, W. Quelle courbe est égale à sa developpée? Un cas simple. Simon Stevin **30**, 17–24 (1954).

The authors exhibit explicit equations for a class of curves in the plane which have the property that they are congruent to their evolutes. An error in a paper by Puiseux [J. Math. Pures Appl. **9**, 377–399 (1844)] on a generalization of this problem is noted and corrected.

S. B. Jackson (College Park, Md.).

Kurita, Minoru. Generalized evolute in Klein spaces. J. Math. Soc. Japan **5**, 355–364 (1953).

The enveloping theorem of an evolute of a curve on the euclidean plane is generalized for the case of figures in Klein spaces by the method of moving frames of E. Cartan. The idea of the paper is the same as in Nagoya Math. J. **1**, 19–23 (1950) [these Rev. **12**, 744]. In addition the process is given for the obtaining of theorems in Klein spaces analogous to Euler-Savary's theorem on the euclidean plane.

J. A. Schouten (Epe).

Morduhai-Boltovskoi, D. Geodesic lines of the ellipsoid in non-Euclidean space. Doklady Akad. Nauk SSSR (N.S.) **94**, 991–993 (1954). (Russian)

Let δ be the distance from the origin to planes tangent to an ellipsoid along a geodesic, and r the semidiameters through the center of the ellipsoid parallel to the tangents to the geodesic. A theorem of Joachimsthal [J. Reine Angew. Math. **26**, 155–171 (1843)] states that, along a fixed geodesic, $\delta r=$constant. The author observes that, if the semidiameter is specified to be perpendicular to the plane which passes through the center and is perpendicular to the geodesic, then the theorem has meaning in hyperbolic space. Representing the ellipsoid as the locus $Ax^2+By^2+Cz^2=u^2$ in the space $x^2+y^2+z^2-u^2=1$, it is found that Joachimsthal's proof carries over without essential change. The result, however, contains hyperbolic functions: $\operatorname{sh}\delta \operatorname{th} r=$constant. It is also pointed out that, by using Hamilton's equations, formulae for the geodesics may be obtained in terms of ultraelliptic integrals, and these equations are given.

L. W. Green (Minneapolis, Minn.).

Bompiani, Enrico. Complessi lineari di piani nello spazio a cinque dimensioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 719–723 (1953).

The paper gives a detailed classification of the linear plane complexes in five-dimensional projective space. This corresponds to the classification of trivectors in six-dimensional affine space.

J. A. Schouten (Epe).

Vranceanu, G. Sur les espaces V_4 ayant comme groupe de stabilité un G_4. Publ. Math. Debrecen **3** (1953), 24–32 (1954).

The group of motions and the stability group of a V_n have at most $\frac{1}{2}n(n+1)$ and $\frac{1}{2}n(n-1)$ parameters respectively. Special results were obtained by Fubini, Egoroff and the author [cf. the reviewer's Ricci calculus, second ed. (to appear), p. 348]. It is proved here that the special symmetrical V_4's of Cartan that have an eight-parameter group of motions are the only V_4's with this property.

J. A. Schouten (Epe).

Singal, M. K., and Ram Behari. Generalization of Codazzi's equations in a sub-space imbedded in a Riemannian manifold. Math. Student **22**, 31–36 (1954).

Etant donné un sous-espace V_n d'un espace Riemannien V_m et un système de $(m-n)$ congruences de courbes de V_m tel que par chaque point de V_n il passe une courbe et une seule de chaque congruence, les auteurs généralisent des résultats antérieurs de Mishra [Ganita **3**, 37–40 (1952); ces Rev. **14**, 581] relatifs au cas où $m=n+1$, et obtiennent des équations invariantes d'où l'on déduit aussitôt les équations de Mainardi-Codazzi pour les hypersurfaces V_n de V_{n+1}, ainsi que les équations analogues pour un sous-espace V_n de V_m, et, plus particulièrement, les résultats bien connus relatifs à l'espace ordinaire à trois dimensions.

P. Vincensini (Marseille).

Singal, M. K., and Ram Behari. On the totally geodesic sub-spaces imbedded in a Riemannian space. Math. Student **22**, 37–41 (1954).

Suivant un résultat dû à Ricci [Atti Ist. Veneto Sci. Lett. Arti (8) **6**(63), 1233–1239 (1904)], si un espace Riemannien V_{n+1}, admet une hypersurface V_n totalement géodésique, les normales à V_n en chaque point sont des directions principales (de Ricci) pour V_{n+1}. Les auteurs généralisent cette proposition à un sous-espace V_n plongé dans un espace de Riemann V_m, et montrent que si V_n est un sous-espace totalement géodésique de V_m ses normales en chaque point sont directions principales pour V_m.

P. Vincensini.

Takano, Kazuo, and Imai, Tyuiti. Note on the conformal theory of subspaces. Tensor (N.S.) **3**, 108–118 (1954).

Let V_m, $\bar{V}_m$ be conformally equivalent Riemann spaces whose corresponding metric tensors $g_{\alpha\beta}$, $\bar{g}_{\alpha\beta}$ in a coordinate system $\{y^\alpha\}$ transform according to $\bar{g}_{\alpha\beta}=e^{2\sigma}g_{\alpha\beta}$ and let V_n, $\bar{V}_n$ $(n<m)$ with metric tensors g_{ij}, $\bar{g}_{ij}$ in a coordinate system $\{x^i\}$ be corresponding subspaces of V_m, $\bar{V}_m$ respectively. The authors prove the theorem: (A) The coefficients of connection

$$\Pi^\alpha{}_{\beta\gamma}=\left\{{\alpha \atop \beta\gamma}\right\}+\delta_\beta{}^\alpha p_\gamma+\delta_\gamma{}^\alpha p_\beta-g_{\beta\gamma}g^{\alpha\delta}p_\delta$$

and

$$\Pi^i_{jk}=\left\{{i \atop jk}\right\}+\delta_j{}^ip_k+\delta_k{}^ip_j-g_{jk}g^{il}p_l$$

are conformally invariant if p_α is a vector satisfying $p_\alpha-\bar{p}_\alpha=\partial\sigma/\partial y^\alpha$ and $p_i=p_\alpha\partial y^\alpha/\partial x^i$. They then show that the conformal coefficients of connection defined by the reviewer [Trans. Amer. Math. Soc. **56**, 309–433 (1944); these Rev. **6**, 105] may be obtained by a suitable specialization of p_α in (A). They also give another proof of the conformal character of the derivation tensor E_{ij} first defined by the reviewer. The proof of (A) given in this paper depends upon the use of the conformal tensor density of T. Y. Thomas and the conformal coefficients of connection of J. M. Thomas. The same result may also be obtained using a proof given in the cited paper of the reviewer. [On p. 333 delete the special definition of η_α given in (8.10) there and let η_α (identified with p_α of the present paper) be any vector satisfying (8.9). The rest of the reviewer's proof is unchanged.]

A. Fialkow (Brooklyn, N. Y.).

Tonooka, Keinosuke. Theory of subspaces in a geometry based on a multiple integral. I. Metric tensor and theory of connections. Tensor (N.S.) **3**, 75–83 (1954).

In an n-dimensional manifold X_n with point coordinates $x^1, \cdots, x^n$ a k-dimensional surface S_k is defined by the parametric equations $x^i=x^i(u^\alpha)$, $i=1, \cdots, n$, $\alpha=1, \cdots, k$. The

quantities n^i, $p_\alpha{}^i=\partial x^i/\partial u^\alpha$, $p^i{}_{\alpha\beta}=\partial^2x^i/\partial u^\alpha\partial u^\beta$ determine the surface element of the second order of S_k. The manifold of all k-dimensional surface elements of the second order is denoted by $X_n{}^{(2)}$. The author considers an intermediate surface $S_m(n>m>k)$ of X_n given parametrically by $x^i=x^i(y^a)$, $a, b=1, \cdots, m$, in which the S_k, given by $y^a=y^a(u^\alpha)$, is immersed so that the y^a, $q_\alpha{}^a=\partial y^a/\partial u^\alpha$, $q^a{}_{\alpha\beta}=\partial^2y^a/\partial u^\alpha\partial u^\beta$ are elements of a space $X_m{}^{(2)}$. The $X_m{}^{(2)}$ and $X_n{}^{(2)}$ are related by $x^i=x^i(y^a)$,

$$p_\alpha{}^i=B_a{}^iq_\alpha{}^a,\quad p^i{}_{\alpha\beta}=B_a{}^iq^a{}_{\alpha\beta}+B^i{}_{ab}q_\alpha{}^aq_\beta{}^b,$$

where $B_a{}^i=\partial x^i/\partial y^a$, $B^i{}_{ab}=\partial^2x^i/\partial y^a\partial y^b$. This set of relationships is then regarded as constituting a set of parametric equations of $X_m{}^{(2)}$ in $X_n{}^{(2)}$. The metric tensors and connexion parameters of $X_m{}^{(2)}$ and $X_n{}^{(2)}$ are then derived from a fundamental function $F(x^i, p_\alpha{}^i, p^i{}_{\alpha\beta})$ and relations established between them. *E. T. Davies* (Southampton).

Kosambi, D. D. The metric in path-space. Tensor (N.S.) 3, 67–74 (1954).

According to a theorem of Whitney [Ann. of Math. (2) 37, 645–680 (1936)] a neighborhood in path-space may be endowed with a Riemannian metric. But the relation between metric and paths does not seem sufficiently clear. It is proved here that by a suitable projective change of the parameter, the paths of a symmetric affine connection may be made the geodesics. There is another theorem concerning the mapping of paths and straight lines.

J. A. Schouten (Epe).

Rund, Hanno. On the analytical properties of curvature tensors in Finsler spaces. Math. Ann. 127, 82–104 (1954).

The author's approach to the study of Finsler spaces is essentially different from that of most writers on the subject in that he regards Finsler spaces as locally Minkowskian whereas others have followed E. Cartan [Les espaces de Finsler, Hermann, Paris, 1934] in regarding them as locally Euclidean. This paper is a sequel to another [Math. Ann. 125, 1–18 (1952); these Rev. 14, 499] devoted to the curvature of Finsler spaces. The first part of the paper is concerned with the introduction of a suitable operation of covariant derivation with respect to a coordinate n^k. In previous work the author has used coefficients of connection which are suitable for covariant derivation along a curve, but do not lead from a tensor to another with an additional covariant index. He gives a simple geometrical construction leading to connection coefficients with the required properties. In spite of the different approach referred to above the final expressions arrived at by the author for the connection coefficient and written P^* coincide with the functions Γ^* occurring in Cartan's work. In fact, from equations (19) of Cartan and (2.5) of Rund we see that $x'^i\Gamma^h{}_{ij}=\partial G^h/\partial x'^j$ of Cartan coincides with $x'^iP^h{}_{ji}$ of Rund, so that if we introduce the operator

$$X_i=\partial/\partial x^i-(\partial G^l/\partial x'^i)\partial/\partial x'^l$$

we can write

$$X_ig_{jh}+X_jg_{hi}-X_hg_{ij}=2P^*{}_{ij,h}(\mathrm{Rund})=2\Gamma^*{}_{ihj}(\mathrm{Cartan}).$$

The author describes fully the special difficulties arising in the definition of the covariant derivative of a tensor field, involving the directional as well as the positional argument. For such tensor fields he then introduces relative and absolute covariant derivation, with appropriate curvature tensors and identities satisfied by these curvature tensors.

E. T. Davies (Southampton).

Rund, Hanno. The scalar form of Jacobi's equations in the calculus of variations. Ann. Mat. Pura Appl. (4) 35, 183–202 (1953).

The subject matter of this paper consists of topics which were treated in many papers following the classic paper on geodesic deviation in Riemannian spaces by Levi-Civitá [Math. Ann. 97, 291–320 (1926)]. The present author had used the idea of geodesic deviation in his approach to the problem of giving a suitably geometrical definition of curvature in Finsler spaces of two dimensions. Whereas the equations of geodesic deviation had usually been given in a form involving the components of the variation vector, the present treatment concentrates on the equation satisfied by the length of the variation vector. The scalar form for the equations of geodesic deviation are given here for the general n-dimensional case, and there are geometrical interpretations given in the light of the author's own approach to Finsler spaces (as locally Minkowskian spaces). The form given to the equations of geodesic deviation are a special case of the Jacobi equations for the original problem in the calculus of variations. These in turn are the Euler equations for the second variation so that the author is led to a geometrical theory of the accessory problem, and to new interpretations of the curvature tensor. *E. T. Davies.*

Rund, Hanno. Über nicht-holonome allgemeine metrische Geometrie. Math. Nachr. 11, 61–80 (1954).

The author generalizes the results of certain authors, particularly those of Synge [Philos. Trans. Roy. Soc. London. Ser. A. 226, 31–106 (1926)] on non-holonomic Riemannian geometry to the corresponding questions in Finsler geometry. He considers a function $F(x, x')$ which is at the basis of a Finsler geometry and a set of conditions $G_{(\mu)}(x, x')=0$ where the functions $G_\mu(x, x')$ are positive homogeneous of the first order in x'. The extremal curves of the Finsler geometry based upon F are also the autoparallel curves with respect to several possible definitions of parallel transport in the space. If, however, consideration is restricted to curves for whose tangent vectors the condition $G_{(\mu)}=0$ holds, the extremal curves will not in general coincide with autoparallels. If F is regarded as the homogenized Lagrangian function of a dynamical system for which the condition $G_{(\mu)}=0$ is satisfied, then the autoparallels are the path curves of the dynamical system. This paper is mainly concerned with these path curves. After obtaining generalizations of the equations of path curves as given by Synge [loc. cit.], the extremals of the problem of Lagrange associated with the function F and the conditions $G_{(\mu)}=0$ are obtained, and conditions are given in order that these extremals may coincide with the path curves. Analogues of the equations of geodesic deviation are then obtained for neighbouring path curves. *E. T. Davies.*

***Lichnerowicz, A. Equations de Laplace et espaces harmoniques.** Premier colloque sur les équations aux dérivées partielles, Louvain, 1953, pp. 9–23. Georges Thone, Liège; Masson & Cie, Paris, 1954.

This is an expository article on harmonic spaces, i.e. Riemannian spaces in which Laplace's equation admits 'simple' solutions. It includes a description of the original Copson-Ruse definition and an account of all the main results known to date, many of which are due to the author.

A. G. Walker (Liverpool).

Rosenlicht, Maxwell. Simple differentials of second kind on Hodge manifolds. Amer. J. Math. **75**, 621–626 (1953).

Soit V une variété analytique complexe. Une différentielle simple sur V (1-forme différentielle méromorphe) est dite de seconde espèce si, en chaque point, elle est égale à la différentielle d'un germe de fonction méromorphe. Soit Ω_2 l'espace vectoriel des formes de seconde espèce et Ω_e l'espace des différentielles de fonctions méromorphes sur V (differentielles exactes); Ω_e est un sous-espace de Ω_2. La variété V est appelée variété de Hodge si elle est kählérienne compacte et si la 2-forme fondamentale est homologue à un multiple scalaire d'un cycle entier. Théorème: Si V est une variété de Hodge, dim $\Omega_2/\Omega_e = B_1 = 2q$, premier nombre de Betti de V. Démonstration. La dimension de l'espace vectoriel des différentielles de première espèce sur V est q; les intégrales de première espèce définissent une application analytique ϕ de V sur une variété abélienne A de dimension q. Par une translation sur A, on peut transformer toute différentielle de seconde espèce sur A, en une différentielle de la même classe modulo les différentielles exactes, et dont le lieu polaire ne contient pas $\phi(V)$; son image, dans V, est une forme de seconde espèce. Alors, il suffit de démontrer le théorème pour la variété abélienne A. Ce théorème est connu si V est une courbe algébrique; l'auteur en déduit le théorème pour la variété jacobienne d'une courbe tracée sur une variété abélienne simple, puis pour toute variété abélienne, donc pour toute variété de Hodge. Si V est une variété de Hodge, toute différentielle simple de seconde espèce est égale, au voisinage de chaque point, à une différentielle exacte, à l'addition près d'un germe de forme holomorphe. Enfin, si V est kählérienne compacte, toute 1-forme différentielle méromorphe égale, en chaque point, à la différentielle d'un germe de fonction méromorphe, modulo les germes de formes holomorphes, est fermée, donc de seconde espèce. *P. Dolbeault* (Paris).

Patterson, E. M. A characterisation of Kähler manifolds in terms of parallel fields of planes. J. London Math. Soc. **28**, 260–269 (1953).

Let $M=M^n$ be a real differentiable manifold of dimension n whose tangent bundle we denote by $T=T(M'')$, and let $CT=T\otimes_R C$ (R, C the real and complex numbers respectively). For each point $x\in M$ the fiber $(CT)_x$ is a complex vector space of dimension n, and a law which assigns differentiably to every $x\in M$ a (complex) vector subspace of $(CT)_x$ of dimension m is called an m-distribution (or field of complex m-planes).

Given a real torsionless affine connection Γ over M, an m-distribution π_m is parallel with respect to Γ if and only if the components $\lambda^a{}_{(\alpha)}$ $(a=1,\cdots,n;\alpha=1,\cdots,m)$ of the basis vector fields of π_m over each coordinate neighborhood satisfy $\lambda^a_{(\alpha),b}=A^{(\beta)}_{(\alpha)b}\lambda^a_{(\beta)}$, where the comma denotes covariant differentiation with respect to Γ and where the $A^{(\beta)}_{(\alpha)b}$ are the components of local covariant vector fields. If $M=M^n$ $(n=2m)$ is a complex-analytic manifold with Hermitian metric $g=(g_{ab})$, let $h=(h_{ab})$ be the corresponding real skew-symmetric tensor field and write $p=(p_{ab})=g+ih$. The solutions λ^a of $p_{ab}\lambda^b=0$ define an m-distribution π_m, and the author points out that the Hermitian metric is a Kähler metric if and only if π_m is parallel with respect to the canonical connection associated with the given metric. He further states the theorem that M^{2m} admits a complex structure if and only if there exists an m-distribution π_m and connection Γ such that: (i) π_m is parallel with respect to Γ; (ii) π_m and its conjugate distribution $\bar{\pi}_m$ have only the zero vector in common. *D. C. Spencer.*

Wong, Y. C. Fields of parallel planes in affinely connected spaces. Quart. J. Math., Oxford Ser. (2) **4**, 241–253 (1953).

Definitions and conditions for parallel planes, given by the reviewer in the case of a Riemannian space, are here extended to a space of symmetric affine connection. It is shown that there is associated with every parallel plane a "double" tensor $A^\beta{}_{\alpha kl}$, where suffixes α, β relate to the basis in the plane and k, l, relate to the coordinate system. To a certain extent this tensor and its contraction $A_{kl}=A^\alpha{}_{\alpha kl}$ characterise a parallel plane and have properties analogous to those of a curvature tensor. In particular, $A^\beta{}_{\alpha kl}=0$ if and only if the plane is strictly parallel.

A canonical form is found for a space admitting two parallel planes (which may or may not intersect), and the results are applied to a Riemannian space as a special case. Finally, it is shown that a Riemannian V_n (n even) which admits two non-intersecting parallel null $\frac{1}{2}n$-planes admits a metric of Kähler form, a result given independently in the paper reviewed above. *A. G. Walker.*

Guggenheimer, H. Formes et vecteurs pseudo-analytiques. Ann. Mat. Pura Appl. (4) **36**, 223–246 (1954).

On a compact complex manifold V^{2n} the geometric genus p_q is the number of linearly independent skew tensors $\xi_{\alpha_1\cdots\alpha_q}$ whose components are holomorphic; the author generalises this to a "simplectic" manifold as being the corresponding number for tensors $\xi_{\alpha_1\cdots\alpha_q}$ which are pseudo-analytic and no linear combination of which is harmonic-non-pseudo-analytic. He then proves that if $p_n\geqq 1$ and if there exists a pseudo-analytic vector field without singularities then also $p_{n-1}\geqq 1$; this proposition had been established originally by Picard for V^{2n} algebraic and by the reviewer for V^{2n} Kaehler—a similar extension of a theorem of Picard-Bochner for the case $p_n\geqq 2$.

Another theorem obtained is that a Hermitian manifold is Kaehler-of-restricted-type if and only if $g^{r\bar{s}}R_{r\bar{s}}$ is positive definite throughout. The author also extends to pseudo-analytic forms the theorem of Severi-Hodge-Jongmans on the relations between periods of Abelian integrals. All this is done in a context which is of import to the study of complex manifolds in general. *S. Bochner.*

Pidek, H. Sur les objets géométriques de la classe zéro qui admettent une algèbre. Ann. Soc. Polon. Math. **24** (1951), no. 2, 111–128 (1954).

The special objects of class zero with one component (classification of Schouten and Haantjes) are called objects of type R^i if the defining functions are of class C^i (continuous derivations of order i). After the demonstration of a preliminary proposition for the case $n=1$, in §3 a proposition is formulated for two objects of the same kind of type R^1 concerning the special forms the transformation can be written in. The proposition is decomposed into five lemmas. *J. A. Schouten* (Epe).

Izmaĭlov, V. D. On a system of pseudo-tensors on two-dimensional surfaces of affine spaces. Doklady Akad. Nauk SSSR (N.S.) **94**, 9–12 (1954). (Russian)

In a previous paper [same Doklady (N.S.) **85**, 477–480 (1952); these Rev. **14**, 319] the author derived the symmetric tensors γ_{ij} and β_{ijk} on a surface X_2 in an affine flat space E_6 of dimension 6. From the same point of

view he gives in the present paper its generalization that a symmetric tensor of second order g_{ij} and a tensor $B_{a_m\cdots a_1, b_m\cdots b_1, \cdots, h_m\cdots h_1}$ can be found on a surface X_2 in an affine flat space E_n of arbitrary dimension n, such that $N=\frac{1}{2}(m-1)(m+2)>0$, where m is the largest positive integer satisfying the relation $n>\frac{1}{2}(m-1)(m+2)$ and B has just N systems of m indices, being symmetric in every two indices of a system and skew-symmetric in every two systems. In showing this result he deals at first with the cases $N=1$, 2 and then with the cases: (a) N and m are both even or odd at the same time; (b) one of N and m is even and the other odd. The two cases (a) and (b) are entirely different from each other. *A. Kawaguchi* (Sapporo).

De Mira Fernandes, A. Grandezze pseudo-estensoriali nella geometria differenziale d'ordine superiore. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **2**, 361–380 (1952).

Starting from the well known extensors of Craig, the author defines other geometric objects, called pseudo-extensors. Their manner of transformation is more complicated, but they can be put in correspondence to the ordinary co- and contravariant extensors. The definition can be generalized for K-spreads. The relations with A. Kawaguchi's operator S^H are given. *J. A. Schouten* (Epe).

Sasayama, Hiroyoshi. On extensors of fractional grade. Tensor (N.S.) **3**, 101–107 (1954).

In a recent paper [Tensor (N.S.) **3**, 53–64 (1953); these Rev. **15**, 255], the author introduced the idea of a space of line-elements of non-integral order and studied the resulting extensors. The present paper is concerned with the study of a space composed of line elements of non-integral order and those of infinite order. It is shown that the covariant exvector of non-integral order (infinite order) and the contravariant exvector of infinite order (non-integral order) are duals. Again, the author shows how to generate covariant and contravariant exvectors of fractional grade from ordinary vectors. This last result is generalized to tensors of arbitrary order. (The reviewer believes that the definition of line elements of infinite order should be clarified.) *N. Coburn* (Ann Arbor, Mich.).

NUMERICAL AND GRAPHICAL METHODS

Moshman, Jack. The generation of pseudo-random numbers on a decimal calculator. J. Assoc. Computing Mach. **1**, 88–91 (1954).

Given a high speed, general purpose, decimal, digital automatic computer, what is a practical way of generating a large number of random digits as a part of a much larger program? The author adopts a method based on the theory of the binomial congruence modulo 10^n where the word length of the machine is n decimal digits. This is the decimal analogue of a method suggested by O. Taussky-Todd for binary machines and consists in forming C_i by the recurrence $C_i\equiv 7\cdot{}_kC_{i-1}$ (mod 10^n) where $0<C_i<10^n$, $k=5$, 9, 13, or 17. The author takes $C_0=1$. Theorems giving the period length of the sequence are proved. Taking $n=11$ and using the UNIVAC, a sequence of 10000 eleven digit decimal numbers were subjected to 5 different frequency tests having to do with occurrences of digits in a fixed digit position. The leading digit positions give good results but by the time the sixth position is reached, the period becomes too short (only 5000) and the results are too uniform. This defect in the method can be avoided by using a slightly different modulus like $10^n\pm a$ where a is a small positive integer. *D. H. Lehmer* (Berkeley, Calif.).

Blanch, Gertrude. On modified divided differences. II. Math. Tables and Other Aids to Computation **8**, 67–75 (1954).

The author defines modified divided differences in the first part of this paper [same journal **8**, 1–11 (1954); these Rev. **15**, 560]. In this part she considers "errors of Type (c)" which are small, unsystematic errors due to rounding or other causes. The main result is a generalization of one due to Lowan and Laderman [Ann. Math. Statistics **10**, 360–364 (1939); these Rev. **1**, 125], giving the distribution of the nth modified difference of numbers possessing rectangular distributions. *A. S. Householder* (Oak Ridge, Tenn.).

Weydanz, W. Eine verbesserte Näherungsgleichung für den Ellipsenumfang. Z. Angew. Math. Mech. **34**, 194–195 (1954).

Aparo, Enzo. Un procedimento iterativo per la risoluzione numerica delle equazioni algebriche. Ricerca Sci. **24**, 1003–1005 (1954).

A long-division process is described for the determination of linear or quadratic factors of polynomials in one variable [cf. Lin, J. Math. Physics **20**, 231–242 (1941); these Rev. **3**, 153]. *E. Frank* (Chicago, Ill.).

Kay, Alan F. On roots of transcendental equations. J. Appl. Phys. **25**, 811 (1954).

Kiss, I. Über eine Verallgemeinerung des Newtonschen Näherungsverfahrens. Z. Angew. Math. Mech. **34**, 68–69 (1954).

A generalization of Newton's formula for the roots of an equation $f(x)=0$ is shown. As a special case, the formula shown by Bodewig [same Z. **29**, 44–51 (1949); these Rev. **10**, 573] is obtained. The formulas found here are applied to the equation $x^n-A=0$. *E. Frank* (Chicago, Ill.).

Collatz, L. Das vereinfachte Newtonsche Verfahren bei algebraischen und transzendenten Gleichungen. Z. Angew. Math. Mech. **34**, 70–71 (1954).

For the Newton formula for the computation of a root of $f(z)=0$ an estimate of error is set up which is simpler than the ordinarily used formula since one only needs $f'(z)$ and not $f''(z)$. Likewise for algebraic equations a simple estimate of error is obtained with the use of Horner's method. *E. Frank* (Chicago, Ill.).

Price, P. C. Gauss's formula of numerical integration and the design of experiments. Proc. Cambridge Philos. Soc. **50**, 491–494 (1954).

The author considers a situation in which the variances of the observations or at least their ratios can be fixed in advance. He then proposes to fix the variances in a certain way and to estimate regression coefficients by minimizing a sum of squares of deviations from the regression values weighted by the inverses of the variances. The author justifies his particular choice of the variances by the following statement. "The object of the method is to simplify the

computations; the alternative criterion for experimental design, the minimization of the standard errors of the coefficients for a given expenditure of time or effort, cannot of course be simultaneously applied." *H. B. Mann.*

Serbin, H. Numerical quadrature of some improper integrals. Quart. Appl. Math. **12**, 188–194 (1954).

Derivations are given for several formulas applicable to the solution of problems in potential theory by numerical methods. These formulas are shown to be related to Multhopp's solution [Luftfahrtforschung **15**, 153–169 (1938)] of the equation of the lifting line in aerodynamics. An example is given of the use of one of these formulas for an aerodynamical problem. *S. Levy* (Washington, D. C.).

Mikeladze, Š. E. On approximate solution of Cauchy's problem. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 245–249 (1954). (Russian)

Consider

$$\partial^n u/\partial t^n = F(t, x_1, \cdots, x_p, u, \cdots, \partial^m u/\partial t^{\alpha_0}\partial x_1^{\alpha_1}\cdots\partial x_p^{\alpha_p}),$$

where $m \leqq n$, $\alpha_0 < n$, with initial conditions

$$\partial^k u/\partial t^k = \varphi_k(x_1, \cdots, x_p), \quad k=0, 1, \cdots, n-1,$$

in some p-dimensional region of the hyperplane $t=0$. Here F and φ_k are assumed to have whatever differentiability and other properties are required to assure existence of a unique solution u and to permit the following manipulations. To compute an approximate solution the author first applies Taylor's theorem to find at $t=rh$, $r=1, 2, \cdots$, $\partial^k u/\partial t^k = \sum_{\lambda=0}^{n-k-1}(rh)^\lambda \varphi_{k+\lambda}/\lambda! + R_k$, where

$$R_k = h^{n-k}\int_0^r (r-t)^{n-k-1}F(th, x, \cdots)dt/(n-k-1)!.$$

The integrals R_k are computed approximately by closed Newton-Cotes quadrature formulas. The equations for $t=rh$ require values of $F(\nu h, \cdots)$ for $0 \leqq \nu \leqq r-1$, which can be calculated from earlier results if $r>2$. To start evaluating $\partial^k u/\partial t^k$ one needs F for $\nu=0$, obtained from the initial data; and for $\nu=1$, obtained for example from Taylor's series to terms of high enough order, say h^k. Finally, the partial differentiations with respect to $x_1, \cdots, x_p$ involved in F are to be performed numerically by central-difference approximations. The procedure is illustrated formally for the p-dimensional wave equation $\partial^2u/\partial t^2 = c^2\Delta u$ at some length, especially for $p=1$ and 2. No numerical examples are discussed. *J. H. Giese* (Havre de Grace, Md.).

Malavard, Lucien, et Boscher, Jean. Applications de la méthode des réseaux superposés à l'étude de divers problèmes d'élasticité. C. R. Acad. Sci. Paris **238**, 1093–1095 (1954).

The use of two networks with connected nodes which has been used to solve the biharmonic difference equation [Bull. Soc. Franç. Méc. **2**, no. 8, 43–57 (1953)] is extended to the solution of difference equations arising from a class of pairs of linear second-order partial differential equations in two functions. A number of applications to problems in elasticity is given. *C. Saltzer* (Cleveland, Ohio).

Müller, Max. Über die Konvergenz eines Verfahrens zur Berechnung der Fourier-Koeffizienten. Math. Z. **60**, 81–87 (1954).

The author proves the convergence (with precise error estimates) of a simple process for determining the Fourier coefficients of a function $f(x)$ of period ω. The function sign(u) is used, where

$$\begin{aligned}\operatorname{sign}(u) &= 1 \quad \text{for} \quad u>0,\\ &=0 \quad \text{for} \quad u=0,\\ &=-1 \quad \text{for} \quad u<0.\end{aligned}$$

With this convention the step-functions

$$C_n(x) = \operatorname{sign}\{\sin(2n\pi x/\omega)\} \text{ and } S_n(x) = \operatorname{sign}\{\sin(2n\pi x/\omega)\}$$

are introduced. By formally inverting an infinite system of simultaneous linear equations, the Fourier coefficients of $f(x)$ are obtained in terms of the numbers γ_n and σ_n defined by $\int_0^\omega f(x)C_n(x)dx = 2\omega\gamma_n/\pi$, $\int_0^\omega f(x)S_n(x)dx = 2\omega\sigma_n/\pi$. The author explains that the series thus obtained for the Fourier coefficients of $f(x)$ in terms of γ_n and σ_n have been used for some time; but he presents the apparently missing proof of convergence. Reference is made to the application of this method for the harmonic analysis of a function $f(x)$ given by a graph. *E. Isaacson* (New York, N. Y.).

Have

***Faster than thought. A symposium on digital computing machines.** Edited by B. V. Bowden. Sir Isaac Pitman & Sons, Ltd., London, 1953. xix+416 pp. (19 plates, 2 insets). 35 shillings.

The editor is an engineer connected with a company that has played a prominant role in the machine development field in England. The book is actually a symposium containing articles by 24 people in the field, essentially all of whom are English. The text itself consists of three main parts. The first part contains a very brief history of computation centering mainly about the work of Babbage. It also contains some material on circuitry, organization, and allied topics of interest to machine designers as well as a small amount of information on programming. The second part is devoted to a discussion of particular machines in England and America. It contains a detailed discussion of British machines; however, a quite negligible portion is devoted to American developments. The third part is devoted to some applications of computing machines. In this part there is some discussion of crystallography, ballistics, and a number of other topics.

The book has the same strengths and weaknesses present in all scientific symposia. It covers a substantial number of topics but with varying emphases and depths of insight. This is perhaps inevitable in a work which attempts to be compendious. The author of the chapter on applications to meteorology is apparently unaware of the fundamental work in the field by Charney, Phillips, von Neumann and others. In general, the bibliographic material in the book is quite weak. The book will undoubtedly be welcomed by many persons desiring to learn generally about the work going on in England. The reader will find many interesting anecdotes and historical comments scattered throughout the text.

H. H. Goldstine (Princeton, N. J.).

Heinhold, J. Integriermaschinen mit nicht beschränkten Varianzbereichen. Z. Angew. Math. Mech. **34**, 64–65 (1954).

Clippinger, R. F., Dimsdale, B., and Levin, J. H. Automatic digital computers in industrial research. III. J. Soc. Indust. Appl. Math. **2**, 36–56 (1954).

For parts I and II see same J. **1**, 1–15, 91–110 (1953); these Rev. **15**, 167, 474.

Jaeger, J. C., Clarke, J. D., and Oliver, A. R. A mechanical differential analyser in the University of Tasmania. Australian J. Sci. **16**, 185–188 (1954).

Wilkes, M. V. Can machines think? Proc. I. R. E. 41, 1230–1234 (1953).

Shannon, Claude E. Computers and automata. Proc. I. R. E. 41, 1234–1241 (1953).

Thomas, Walker H. Fundamentals of digital computer programming. Proc. I. R. E. 41, 1245–1249 (1953).

Hopper, Grace M., and Mauchly, John W. Influence of programming techniques on the design of computers. Proc. I. R. E. 41, 1250–1254 (1953).

Rubinoff, Morris. Analogue vs. digital computers—a comparison. Proc. I. R. E. 41, 1254–1262 (1953).

Buchholz, Werner. The system design of the IBM Type 701 Computer. Proc. I. R. E. 41, 1262–1275 (1953).

Frizzell, Clarence E. Engineering description of the IBM Type 701 Computer. Proc. I. R. E. 41, 1275–1287 (1953).

Ross, Harold D., Jr. The arithmetic element of the IBM Type 701 Computer. Proc. I. R. E. 41, 1287–1294 (1953).

Huskey, H. D., Thorensen, R., Ambrosio, B. F., and Yowell, E. C. The SWAC-design features and operating experience. Proc. I. R. E. 41, 1294–1299 (1953).

Greenwald, Sidney, Haueter, R. C., and Alexander, S. N. SEAC. Proc. I. R. E. 41, 1300–1313 (1953).

Sherertz, Paul C. Electronic circuits of the NAREC Computer. Proc. I. R. E. 41, 1313–1320 (1953).

Wheeler, David J., and Robertson, James E. Diagnostic programs for the ILLIAC. Proc. I. R. E. 41, 1320–1325 (1953).

Erickson, R. S. The logistics computer. Proc. I. R. E. 41, 1325–1332 (1953).

Crosman, Loring P. The Remington Rand Type 409-2 Electronic Computer. Proc. I. R. E. 41, 1332–1340 (1953).

Shannon, Claude E., and Moore, Edward F. Machine aid for switching circuit design. Proc. I. R. E. 41, 1348–1351 (1953).

Palevsky, Max. The design of the Bendix Digital Differential Analyzer. Proc. I. R. E. 41, 1352–1356 (1953).

Serrell, Robert. Elements of Boolean algebra for the study of information-handling systems. Proc. I. R. E. 41, 1366–1380 (1953).

Elbourn, Robert D., and Witt, Richard P. Dynamic circuit techniques used in SEAC and DYSEAC. Proc. I. R. E. 41, 1380–1387 (1953).

Gluck, S. E., Gray, H. J., Jr., Leondes, C. T., and Rubinoff, M. The design of logical or-and-or pyramids for digital computers. Proc. I. R. E. 41, 1388–1392 (1953).

Eckert, J. P., Jr. A survey of digital computer memory systems. Proc. I. R. E. 41, 1393–1406 (1953).

Craven, T. L. More about circuits and logic. Electronic Engrg. 26, 302–305 (1954).

Ware, Willis H. The logical principles of a new kind of binary counter. Proc. I. R. E. 41, 1429–1437 (1953).

White, Garland S. Coded decimal number systems for digital computers. Proc. I. R. E. 41, 1450–1452 (1953).

Burke, Harry E., Jr. A survey of analog-to-digital converters. Proc. I. R. E. 41, 1455–1462 (1953).

Gray, H. J., Jr., Levonian, P. V., and Rubinoff, M. An analog-to-digital converter for serial computing machines. Proc. I. R. E. 41, 1462–1465 (1953).

Spero, Robert E. Effectiveness of two-step smoothing in digital control computers. Proc. I. R. E. 41, 1465–1469 (1953).

McCool, William A. An AM-FM electronic analog multiplier. Proc. I. R. E. 41, 1470–1477 (1953).

Howe, Robert M., and Haneman, Vincent S., Jr. The solution of partial differential equations by difference methods using the electronic differential analyzer. Proc. I. R. E. 41, 1497–1508 (1953).

Kulikov, D. K. Mechanization of astronomical computations. Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. 5, 512–545 (1953). (Russian)

***Pentkowski, M. W. Nomographie.** Akademie-Verlag, Berlin, 1953. xv+268 pp. DM 15.00.

Translation of the author's "Nomografiya" [Gostehizdat, Moscow-Leningrad, 1949; these Rev. 13, 78].

***Meyer zur Capellen, W. Leitfaden der Nomographie.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953. ii+178 pp. DM 17.40.

Have

An excellent manual of the principal methods for the representation of formulas by slide rules and charts, profusely illustrated by examples and exercises. The first 92 pages are concerned with theory and the next 78 with the practical construction of charts. Finally, there is a table of about 75 types of formulas treated in the book, and a brief bibliography with mostly German titles. Although written primarily for the engineer, chemist, etc. with applications in mind, the theoretical aspects are treated adequately and at a somewhat more advanced level than is the case in recent books in English. *P. W. Ketchum* (Urbana, Ill.).

Richter, W. Koordinatentransformationen mit Hilfe eines Fluchtliniennomogramms und Anwendungen auf die graphische Lösung von Differentialgleichungen. Österreich. Ing.-Arch. 8, 39–47 (1954).

The representation of a transformation of coördinates in a plane by a pair of nomograms or other charts is studied, in particular, inversion in a circle, collineations, affine transformations, polar coordinates. Applications to the solution of certain types of differential equations by a method previously given by the author [Z. Angew. Math. Mech. 32, 120–129 (1952); these Rev. 14, 323] are proposed.

P. W. Ketchum (Urbana, Ill.).

Athen, H. Physikalische Darstellung mathematischer Funktionen. Math. Naturwiss. Unterricht 7, 9–14 (1954).

ASTRONOMY

Bragard, Lucien. Sur la déviation de la verticale en un point du cogéoïde. C. R. Acad. Sci. Paris **237**, 380–382 (1953).

New formulae for the N-S and E-W components of the deviation of the vertical at a point of a cogeoid [Bragard, these Rev. **13**, 167; **14**, 94, 271, 414, 415] are derived. The contributions of the gravity anomalies on the sphere of reference are included in these formulae. The latter require the computation of variations of $\Delta r_1(P) = r(P') - R(P'')$ along the meridian and the parallel at a point P of the sphere, P' being the point of the cogeoid and P'' that of a figure of reference having the same volume.

W. S. Jardetzky (New York, N. Y.).

Bragard, Lucien. Une simplification de la formule fondamentale de la géodésie dynamique. Bull. Géodésique 1953, 139–151 (1953). (German, English, Spanish, Italian summaries)

From the equality of the volumes (and hence of the masses) of the cogeoid and the figure of reference, and from the coincidence of their centres of gravity, two relations can be derived satisfying the gravity anomalies Δg, supposed distributed over the sphere of unit radius. These two relations allow us to simplify Stokes's formula. (From the author's summary.) *W. S. Jardetzky* (New York, N. Y.).

Evrard, Léon. Sur la figure piriforme. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 130–138 (1954).

The analysis of the author's previous paper [Ann. Astrophysique **14**, 17–39 (1951)] concerning the equilibrium shape of double stars is extended to take into account terms of the fourth power of the reciprocal of the distance between the stars. The resulting equilibrium figure is found to be very similar to the classical pear-shaped one derived by Poincaré. *R. G. Langebartel* (Urbana, Ill.).

***Smart, W. M. Celestial mechanics.** Longmans, Green and Co., London-New York-Toronto, 1954. vii+381 pp. $13.50.

Chapter headings: The elliptic orbit. Expansions for an elliptic orbit. The general equations of motion and their known integrals. Lagrange's planetary equations. Outline of the solution of Lagrange's planetary equations. Expansion of the disturbing function. Canonic equations. The canonic constants for the elliptic orbit. Contact transformations. The Delaunay and Poincaré variables. Delaunay's lunar theory. Secular inequalities. Gauss' method for the calculation of secular inequalities. The influence of a resisting medium and the motion of Mercury's perihelion. The discovery of Neptune. De Pontécoulant's lunar theory. The Hill-Brown lunar theory. Secular acceleration of the Moon. Precession and nutation.

This book, according to the Preface, is designed as a text for an introductory course in celestial mechanics. As such it departs considerably in subject-matter emphasis from most previous texts. While the standard procedure for the determination of an elliptic orbit ephemeris is presented, there is no treatment given of parabolic orbits or of orbit determination from observational data. The great bulk of the book is devoted to the perturbation problem developed for express application to lunar and planetary theory. The three-body problem enters only in this context—exact solutions of special cases, "restricted problem", etc., are not to be found, although Lagrange's equilateral solution is mentioned (but not derived) in connection with the stability of the Trojan asteroids about the equilibrium positions. The author gives a most interesting and detailed comparison of the methods used by Adams and by Leverrier to compute the position of Neptune from the observed deviations in Uranus' orbit. No mention, however, is made of the analogous computation made by P. Lowell in connection with the search for Pluto. The author proves the Virial Theorem but unaccountably does not call it by name, a rather unfortunate oversight in view of the increased use which this proposition has recently found in stellar dynamics and stellar structure theory.

There are two factors that will probably serve to limit its use as a textbook: its complete lack of problems, and its rather stiff price. It seems more likely that the book will find its place as a reference work and as such will prove especially important for its admirable exposition of perturbation theory. *R. G. Langebartel* (Urbana, Ill.).

De Caro, E. Sulla variazione degli elementi orbitali nel moto relativo di une astri di masse variabili. Atti Accad. Gioenia Catania (6) **8** (1951–1952), 141–158 (1953).

The author makes use of the hodograph method for the two-body problem with masses varying exponentially with the time to determine the elements of the orbit. He applies the results to the theory of the origin of the solar system to explain the great distances of the major planets from the sun. *R. G. Langebartel* (Urbana, Ill.).

Popović, Božidar. Les équations nouvelles des perturbations dans le mouvement des planètes. Bull. Acad. Serbe Sci. (N.S.) **5**, Cl. Sci. Math. Nat. Sci. Math. **1**, 123–126 (1952).

Using the vector elements **C**, **D**, τ, introduced by Milanković [Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. **6**, 1–70 (1939); these Rev. **11**, 407] in the perturbation theory of planetary motion, and the perturbative force, the author establishes by direct vector methods the equations for the variations of these elements. *E. Leimanis.*

Mihailović, Dobrivoje. L'allegato all'analisi qualitativa delle forme delle orbite nel problema di due corpi. Bull. Soc. Math. Phys. Serbie **4**, no. 3–4, 49–52 (1952). (Serbo-Croatian. Italian summary)

Let $\mathbf{s}$ be the position vector of the particle A of mass m_1 relative to the particle B of mass m_2, and assume with Batyrev [Akad. Nauk SSSR. Astr. Žurnal **26**, 56–59 (1949); these Rev. **10**, 577] that the total mass $M = m_1 + m_2$ of the system varies according to the law $M = 1/(1+\alpha t)$, $\alpha > 0$. If one introduces the position vector $\mathbf{r}$ of the auxiliary particle P and the pseudo-time τ, defined by $\mathbf{r} = \mathbf{s}/(1+\alpha t)$ and $\tau = 1/\alpha(1+\alpha t)$, then the equation of motion of A reduces to that of P in the classical two-body problem. The author shows that (i) the areal velocity of P along a conic section and that of A along the corresponding spiral-like orbit are equal in absolute values, and (ii) the senses of description of the two corresponding orbits are opposite.

E. Leimanis (Vancouver, B. C.).

Mihailović, Dobrivoje. Application of vector elements to the solution of the problem of two bodies with variable sum of masses. Bull. Soc. Math. Phys. Serbie **5**, no. 3–4, 93–109 (1953). (Serbo-Croatian. English summary)

The vector elements of P [see the preceding review] used by the author are those of Milanković [Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. **6**, 1–70 (1939); these Rev. **11**, 407]: (i) the constant $\mathbf{C}$ in the integral of the areal velocity, (ii) the constant $\mathbf{D}$ in Hamilton's integral, and (iii) the constant T in Kepler's integral—all together six independent scalar constants since $\mathbf{C}\cdot\mathbf{D}=0$ [cf. also E. A. Milne, Vectorial mechanics, Interscience, New York, 1948, pp. 235–241; these Rev. **10**, 488]. Milanković applied this system of elements to elliptic motion only; in this paper the author extends the method to hyperbolic and parabolic motions. *E. Leimanis* (Vancouver, B. C.).

Mihailovitsch, Dobrivoje. Bemerkung über das Jacobische Integral im asteroidischen Dreikörperproblem für den Zufall der elliptischen Bahn des störenden Körpers. Bull. Soc. Math. Phys. Serbie **3**, nos. 3–4, 61–65 (1951). (Serbo-Croatian. German summary)

The Jacobian integral for the Wilkens case of the restricted three-body problem [Festschrift für H. von Seeliger, Springer, Berlin, 1924, pp. 153–168] is derived by vector methods instead of conventional methods. *E. Leimanis.*

Kurth, Rudolf. Das Anfangswertproblem der Stellardynamik. Z. Astrophys. **30**, 213–229 (1952).

The equations of stellar dynamics relate the distribution function $f(x_i, u_i, t)$, the density $D(x_i, t)$, and the gravitational potential $V(x_i, t)$ through the equations

$$\frac{\partial f}{\partial t}+u_i\frac{\partial f}{\partial x_i}-\frac{\partial f}{\partial u_i}\frac{\partial V}{\partial x_i}=0 \quad \text{(Liouville's equation)},$$

$$D=\int\int\int f\,du_1du_2du_3$$

(the integration over the accessible velocity space; definition of density) and

$$\nabla^2 V=4\pi D \quad \text{(Poisson's equation)}.$$

The author asks: Do these equations admit a unique solution such that, at $t=0$, $f(x_i, u_i, 0)$ is some known function $F(x_i, u_i)$? The answer he gives is summarized by the following theorem: Using the given distribution $F(x_i, u_i)$, first determine the density D, and then evaluate in terms of D the gravitational potential V. Consider the motion of the individual stars in this field V. Any arbitrary function of the first integrals of the resulting equations of motion provide a solution of the basic equations for sufficiently small t. The theorem provides the basis for an iteration method of numerically integrating the equations of stellar dynamics.
S. Chandrasekhar (Williams Bay, Wis.).

MECHANICS

✱Gouyon, R. Le problème de mécanique rationnelle à l'agrégation. Librairie Vuibert, Paris, 1954. 256 pp. 2000 francs.

A collection of the problems, followed by solutions, in rational mechanics from the "agrégation masculine", 1932–1939, 1946–1952, and the "agrégation féminine", 1948, 1951. The problems are preceded by four notes on some topics in mechanics.

✱Golubev, V. V. Lekcii po integrirovaniyu uravneniĭ dviženiya tyaželogo tverdogo tela okolo nepodvižnoĭ točki. [Lectures on the integration of the equations of motion of a heavy rigid body about a fixed point.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 287 pp. 6.15 rubles.

The reader of this book will face a mystery. At the turn of the century some of the best minds were fascinated by the question: When do three algebraic integrals exist in the problem of the title? In 1888 Sophie Kowalevski [(K): Acta Math. **12**, 177–232 (1889)] added a third case [$A=B=2C$ (lingam) with equatorial mass center] to those of Euler and Lagrange, and obtained the Bordin prize for it. In 1897 Liouville [Acta Math. **20**, 239–284 (1896)] won the same prize for a paper in which he conjectured that $A=B=C/2n$, n a positive integer, and the center equatorial, is necessary and sufficient for a third algebraic integral. This conjecture still is given as the last word on the subject in the staple treatises. But the book under review mentions rather casually, that Liouville, Husson, and Burgatti (no references given) proved that $n=1$. A reference to Husson [(H): Ann. Fac. Sci. Univ. Toulouse (2) **8**, 73–152 (1906)] will be found by the reader in Whittaker's "Analytical dynamics" [4th ed., Cambridge, 1937, p. 166], unmarred by any comments. On turning to the 80-page paper he will find a maze of cases and subcases, two methods of proof, criticisms of Liouville, and expressions of deep obligation to Appell. In the latter's "Traité de mécanique rationnelle", vol. 2, 4e éd., p. 202 [Gauthier-Villars, Paris, 1924] there is not even a footnote reference to (H). There this reviewer lost the trail.

The book is essentially about (K) with some attention given to the Appelrot case (see below). All the other material (72 pp.) is standard in advanced treatises. The author concentrates so strongly that the steady motions at constant velocity (usually credited to Staude) are barely mentioned. About 100 pages are given to mathematical preliminaries to the quadratures in (K): algebraic functions, Abelian integrals, Riemann surfaces, theta functions, etc. The original presentation in (K) begged for improvement but the author remained unmoved. On the contrary, he contributed to the avoidable maze of variable changes, introduced more notation ill-advisedly, changed (or failed to change) some other notation injudiciously, and created some strange chaos on p. 128. In two places, however, improvements on the (K) proofs could be noticed. The following simple essence of (K) is not easy to glean from the book (different notation is used).

Modifying a Liapounoff proof (development in powers of a small parameter), the book presents the (K) theorem that all solutions are single-valued only in the three classical cases. In all these cases there also exist three algebraic integrals [cf. Whittaker, loc. cit., p. 166]. Since the last multiplier of the equations is known (it is 1), the (K) problem reduces to quadratures. Let h, k, l be the constants of integration, f a body constant, and $x_{1,2}=p\pm iq$. Let

$$R(x_1, x_2)=f^4-k^4+2lf^2(x_1+x_2)+6h^2x_1x_2-x_1^2x_2^2,$$
$$R(x, x)=R(x),$$
$$(x_1-x_2)^2Q=(x_1-x_2)^4w^2-2(x_1-x_2)^2R(x_1, x_2)w$$
$$+R^2(x_1, x_2)-R(x_1)R(x_2).$$

Then $(\partial Q/\partial x_1)^2/R(x_2)=\phi(w)$, a third-degree polynomial in w. Let $w_{1,2}$ be the roots of $Q=0$, and let $e_{1,2}=\pm k$. Then

$$\sum_i dw_i/[\phi(w_i)(w_i^2-k^4)]^{1/2}=0,$$

$$dx_1/[R(x_1)]^{1/2}\mp dx_2[R(x_2)]^{1/2}=\pm i[w^2_{1,2}-k^4]^{1/2}/(w_1-w_2).$$

After this noble sport of hunting for a closed solution is done, the quarry turns out to be a white elephant of the first magnitude. There is some minor game too: "the particular first algebraic integrals". Some of them are subcases of (K): $k=0$ (Delone); $w_1=k$, and ϕ has a double root; etc. There is a case of Bobylev-Steklov: $B=2A$, center on the B axis, $r=0$; and one of Goryachev-Chaplygin: $A=B=4C$, equatorial center, horizontal angular momentum. But the most notable case of a particular third integral is the Hess-Appelrot one: $a^2A(B-C)=c^2C(A-B)$, $b=0$, $Aap+Ccr=0$, where a, b, c are the coordinates of the center. Needless to say, the problem is innocent of any physical content.

A. W. Wundheiler (Chicago, Ill.).

Chrapan, Ján. The Lagrangian rigid body. Mat.-Fyz. Sborník Slovensk. Akad. Vied Umení 2, 23–51 (1952). (Slovak. Russian summary)

The Lagrangian rigid body is a rigid body rotating around a fixed point, it being assumed that the ellipsoid of inertia at the center of rotation is a spheroid whose axis of symmetry passes through the centroid of the body. If Euler's angles are used to describe the position of the body, the equations of motion can be integrated in terms of elliptic functions. The necessary formulas from the theory of elliptic functions are derived in the first three sections of the paper.

The author uses the double subscript notation for theta functions: in this review the notations of Erdélyi et al., "Higher transcendental functions", vol. II, chapter XIII [McGraw-Hill, New York, 1953; these Rev. 15, 419] will be used. The author puts

$$Z_p=\theta_p'(v)/\theta_p(v),\quad \Omega_p=\tfrac{1}{2}\log[\theta_p(\eta-v)/\theta_p(\eta+v)]$$
$$\Pi_p=\eta Z_p+\Omega_p,\quad p=0, 1, 2, 3,$$

[cf. op. cit., p. 363 f.] and gives transformation formulas for Z_p when v is imaginary, or k is imaginary, expresses sums and differences of Z_p, Z_q, gives formulas for Ω_p when v increases by $i\mathbf{K}'$, when k is imaginary, when v is imaginary, formulas for $d\Omega_p/dw$, and similar information for Π_p.

A. Erdélyi (Pasadena, Calif.).

Forbat, N. Sur un problème de mouvement relatif. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 40, 371–376 (1954).

L'équation de mouvement par rapport à $Oxyz$ d'un point matériel libre sous l'action d'une force dérivant d'un potentiel, compte tenu du mouvement donné de $Oxyz$, ne peut généralement pas être intégrée par séparation des variables. L'auteur fait usage d'un théorème donné par lui [même Bull. (5) 30, 462–473 (1946); ces Rev. 8, 102] sur la séparabilité de l'équation de Hamilton-Jacobi. *O. Bottema.*

Weirich, H. Zur Konstruktion des vektoriellen und skalaren Produktes in der räumlichen Mechanik. Z. Angew. Math. Mech. 34, 75–76 (1954).

Capon, R. S. A unified formalism in mechanics. Math. Ann. 127, 305–318 (1954).

In a space with metric tensor a_{ij} a system of curves satisfying the ordinary differential equations

$$(1)\qquad A^p(x^i, \dot{x}^i)=0 \quad (p=1, \cdots, r;\ A^p \text{ homogeneous in } \dot{x}^i)$$

is determined by the condition that the first curvature is stationary. This leads to the differential equations

$$(2)\qquad \ddot{x}^h+\begin{Bmatrix} h \\ ij \end{Bmatrix}\dot{x}^i\dot{x}^j+\lambda_p\partial^h A^p=0,$$

where $\partial^h=\partial/\partial\dot{x}^h$ and the multipliers λ_p can be determined from (1) and (2). As an application the author considers a mechanical system with Lagrange function L depending on q^r, $\dot{q}^r$ and t. Introducing an auxiliary coordinate q^0 and the fundamental form $ds^2=-dq^0dt+L(dt)^2$ it is shown that the natural paths of a mechanical system are given by (2), the $A^p=0$ being the equations of constraint. If the metric is replaced by $ds^2=\frac{1}{2}a_{\infty}^{-1}(dq_0)^2+L(dt)^2$ a set of equations is obtained which may be considered as the differential equations for mechanical-electrical systems of nonholonomic types. The formalism is extended to a Finsler metric.

J. Haantjes (Leiden).

Hydrodynamics, Aerodynamics, Acoustics

Naylor, V. D. The stream function and the velocity potential function. Math. Gaz. 38, 30–34 (1954).

Some expository remarks concerning the connection between the stream function and velocity potential in plane flow of an ideal fluid. *D. Gilbarg* (Stanford, Calif.).

Prem Prakash. Two dimensional steady flows superposable on a source, sink or doublet. Bull. Calcutta Math. Soc. 45, 51–54 (1953).

The author shows that the only plane viscous flows superposable in the sense of Ballabh [Proc. Benares Math. Soc. (N.S.) 2, 69–79 (1940); these Rev. 3, 283] on a plane potential source or doublet must be irrotational.

D. Gilbarg (Stanford, Calif.).

Kito, Fumiki. A note on kinetic energy of fluid motion. Proc. Fac. Eng. Keio Univ. 5 (1952), 67–69 (1953).

The author uses Kelvin's extension of Green's theorem to prove that when a body immersed in water vibrates, the theoretical virtual mass due to vibration is the same whether the water is flowing irrotationally or there is no flow at all.

L. M. Milne-Thomson (Greenwich).

Kito, Fumiki. On vibration of a cylindrical shell immersed in water. Proc. Fac. Eng. Keio Univ. 5 (1952), 32–40 (1953).

Water occupies the region between the walls of two coaxial cylinders. The walls of one cylinder vibrate, those of the other are rigid. Various specialisations of this situation are treated, e.g. when the rigid walls are those of the outer cylinder. The wave equation is approximated to Laplace's equation, which is tantamount to the assumption that the length of the water waves is large compared with the dimensions of the cylinders. Expressions in terms of Bessel's functions are obtained for the virtual mass.

L. M. Milne-Thomson (Greenwich).

Gurevič, M. I., and Haskind, M. D. Jet flow about a contour undergoing a small oscillation. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 599–603 (1953). (Russian)

Envisageons un écoulement plan, permanent avec sillage infini d'un liquide pesant autour d'un contour donné. Ce régime étant supposé déterminé, les auteurs cherchent à

calculer la perturbation qu'il subit lorsque le contour éprouve des vibrations de faible amplitude. La mise en équation du problème repose sur des hypothèses simplificatrices dont les auteurs sont les premiers à souligner l'audace. On est finalement conduit à un problème aux limites curieux dont on peut expliciter la solution dans le cas des vibrations sinusoïdales simples. L'interprétation physique des résultats paraît interessante. *J. Kravtchenko* (Grenoble).

Lavrent'ev, M. A. I. On the theory of long waves. II. A contribution to the theory of long waves. Amer. Math. Soc. Translation no. **102**, 53 pp. (1954).

Translation of Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. **1946**, no. 8, 13–69 (1947) [these Rev. **14**, 102] and reprint of C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 275–277 (1943) [these Rev. **6**, 191].

Haskind, M. D. On wave motions of a heavy fluid. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 15–26 (1954). (Russian)

In the first part of this paper the author considers the motion of a body, S, floating on an infinitely deep fluid with surface waves. Let the velocity potential be

$$\Phi(x, y, z, t) = \varphi(x, y, z)e^{i\sigma t},$$

satisfying the boundary conditions $\phi_z - k\phi = 0$ for $z=0$ $(k=\sigma/g)$, $\phi_n = v_n$ on S, and the asymptotic condition

$$\phi = cR^{-1/2}e^{kz-ikR} + \phi^* \quad (\phi^* = ig\sigma^{-1}r_0e^{kz-ik(x\cos\theta+y\sin\theta)})$$

as $R=(x^2+y^2)^{1/2}\to\infty$. Let $\bar{S}$ be the reflection of S in $z=0$, let $\chi = e^{-kz}H_0^{(2)}(kr)$ ($H_0^{(2)}$ the Hankel function of second kind), and let $f(x, y, z)$ be the function defined by $f_z = \phi_z - k\phi$. The author derives the formula:

$$\phi = f + ke^{kz}\int_\infty^z fe^{-kz}dz + \tfrac{1}{4}kie^{kz}\iint_{S+\bar{S}}(f_n\chi - f\chi_n)dS + \phi^*,$$

and the asymptotic formula for large R;

$$\phi = iM(k, \theta)(k/8\pi R)^{1/2}e^{kz-i(kR-\pi/4)},$$

where

$$M(k, \theta) = \iint_{S+\bar{S}} e^{-kz+ik(x\cos\theta+y\sin\theta)}\{f_n - kf[i\cos\theta\cos(n, x) + i\sin\theta\cos(n, y) - \cos(n, z)]\}dS.$$

Although the first formula requires a knowledge of f throughout the fluid, and the second formula on the boundary S, the author is able to apply the formulas to several cases where this is either known or determined by an integral equation.

In the second part of the paper he considers the two-dimensional wave motion resulting from a submerged source of pulsating strength, first taken stationary and then with a constant horizontal velocity. *J. V. Wehausen.*

Krein, S. G. On functional properties of operators of vector analysis and hydrodynamics. Doklady Akad. Nauk SSSR (N.S.) **93**, 969–972 (1953). (Russian)

Soient: G, un domaine borné, étoilé relativement à l'origine; Γ la frontière de G; H l'espace de Hilbert de vecteurs $\mathbf{V}(x, y, z)$, définis dans G, tels que $\iiint|\mathbf{V}|^2d\tau<\infty$, le produit scalaire $(\mathbf{V}, \mathbf{W})$ de deux éléments de H étant défini au moyen de la formule:

$$(\mathbf{V}\cdot\mathbf{W}) = \iiint \mathbf{V}\cdot\mathbf{W}d\tau;$$

D, un sous espace de H, fermeture de l'ensemble des vecteurs solénoïdaux de H. L'auteur définit des opérateurs convenables, au moyen desquels il construit un élément $\mathbf{W}\in D$, solution du système:

$$\Delta\mathbf{W} = \operatorname{grad} p; \quad \operatorname{div}\mathbf{W} = 0$$

prenant sur Γ (le sens de cette locution étant précisé par l'auteur) les mêmes valeurs qu'un vecteur $\mathbf{f}(x, y, z)$, donné à priori, assez régulier, $\mathbf{f}\in D$. A noter que la scalaire $p(x, y, z)$ est harmonique et que la solution $\mathbf{W}$ est indéfiniment différentiable. De même, l'auteur construit un opérateur convenable sur un autre sous-espace de H. Il peut alors former une solution du système:

$$\Delta\mathbf{V} = \operatorname{grad} p - \mathbf{g}, \quad p = -\frac{1}{4\pi}\iiint_G \frac{\mathbf{g}\cdot\mathbf{r}}{r^3}d\tau.$$

Ces résultats constituent les généralisations des théorèmes d'existence des solutions pour les équations linéaires de l'hydrodynamique des liquides visqueux, dont la première version est du à Odqvist [Math. Z. **32**, 329–375 (1930)]. Les conclusions ci-dessus comportent divers corollaires: (1) l'existence de petits mouvements du liquide visqueux enfermé dans un vase autour d'une position d'équilibre; (2) la justification des procédés variationnels pour le calcul de ces petits mouvements, etc. Un énoncé concernant les petits mouvements autour d'un régime stationnaire complète ce mémoire. *J. Kravtchenko* (Grenoble).

Tyusai
***Aoi, ~~Tadamasa~~. Steady flow of a viscous fluid past a circular cylinder.** Sûributsurigaku kenkyu. Dai 1 kan. Ryutairikigaku no shomondai. I. [Mathematical-physical investigations. Vol. 1. Problems of fluid dynamics. I.] Pp. 130–150. Iwanami shoten, Tokyo, 1950. 480 yen.

Tyusai
***Aoi, ~~Tadamasa~~. Steady flow of a viscous fluid past a sphere.** Sûributsurigaku kenkyu. Dai 1 kan. Ryutairikigaku no shomondai. I. [Mathematical-physical investigations. Vol. 1. Problems of fluid dynamics. I.] Pp. 151–168. Iwanami shoten, Tokyo, 1950. 480 yen.

These two papers give a somewhat more detailed exposition of results in the following two papers by Tomotika and Aoi: Quart. J. Mech. Appl. Math. **3**, 140–161 (1950); Mem. Coll. Sci. Univ. Kyoto. Ser. A. **26**, 9–19 (1950); these Rev. **12**, 59; **13**, 397.

Rumer, Yu. B. Convective diffusion in a heated jet. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 743–744 (1953). (Russian)

L'auteur étudie le problème de la veine noyée s'échappant d'un tube mince dans un liquide de même nature emplissant l'espace. Landau et Lifschitz [Mechanics of continuous media, Gostehizdat, Moscow, 1944] et l'auteur [Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 255–256 (1952); ces Rev. **13**, 791] lui-même ont résolu la question dans l'hypothèse d'un liquide homogène. Ici l'auteur suppose qu'il s'agit d'une solution et il prend en compte le phénomène de diffusion. Il obtient finalement les lois de l'écoulement aux grandes distances de l'orifice des tubes. *J. Kravtchenko.*

Borodin, V. A., and Dityakin, Yu. F. On the stability of plane flows of a viscous fluid between two walls. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 569–578 (1953). (Russian)

On doit à C. C. Lin [Quart. Appl. Math. **3**, 117–142 (1945); ces Rev. **7**, 225] l'étude de la stabilité de l'écoule-

ment laminaire plan d'un liquide visqueux dans un canal à bords rectilignes et parallèles, animé de vitesses parallèles aux parois. Cet auteur ramène la question à la discussion d'une équation différentielle. Les auteurs effectuent cette discussion en utilisant la méthode bien connue de Galerkin dont la validité, en l'espèce, est justifiée en s'appuyant sur certains résultats de Keldych. Les auteurs étudient alors, d'une manière approchée, la stabilité des écoulements dont les profils des vitesses présentent des points d'inflexion: ceci paraît nouveau. Ils concluent à l'instabilité des régimes caractérisés par les profils des vitesses à inflexions dissymétriques. *J. Kravtchenko* (Grenoble).

Stuart, J. T. On the stability of viscous flow between parallel planes in the presence of a co-planar magnetic field. Proc. Roy. Soc. London. Ser. A. **221**, 189–206 (1954).

Linearized equations are derived which govern the stability of viscous flow between two parallel planes in the presence of a co-planar magnetic field. With one approximation (which restricts the range of the Reynolds numbers which can be considered) the author shows that the classical equation of Orr-Sommerfeld is generalized by the addition of a single term involving a non-dimensional parameter q which represents the influence of a magnetic field. The resulting characteristic-value problem in a fourth-order differential equation is discussed by a method similar to those used by Tollmien, Lin and Meksyn in the case when no magnetic field is present. The Reynolds number for neutral stability for a disturbance of assigned wave number is determined for a range of values of the parameter q. The principal result established in the paper is that the critical Reynolds number above which the flow is unstable increases with the strength of the magnetic field.

S. Chandrasekhar (Williams Bay, Wis.).

***Tollmien, Walter. Laminare Grenzschichten.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 11. Hydro- und Aerodynamik, pp. 21–53. Verlag Chemie, Weinheim, 1953. DM 14.00.

This paper is dated June 10, 1947, although it was not published until 1953. It describes the progress in the theory of laminar-boundary layer in Germany during the years 1939–46, with extensive references to the German scientific literature. One third of the article is devoted to the theory of steady boundary-layer flow in an incompressible fluid. This is followed by a short section dealing with the unsteady case. The author then discusses the instability theory of the boundary layer. Finally, there is a section dealing with the steady flow in a boundary layer with variable temperature and variable material properties.

C. C. Lin (Ithaca, N. Y.).

Kaplun, Saul. The role of coordinate systems in boundary-layer theory. Z. Angew. Math. Physik **5**, 111–135 (1954).

It has been noticed by several authors that in solving Prandtl's boundary-layer equations for a semi-infinite plate, if a parabolic, instead of rectangular Cartesian, coordinate system is chosen, the solution will be valid in the whole flow field, singular only at the leading edge. This idea is now fully developed in this paper. The main result is contained in two theorems.

Let $\zeta(\xi, \eta)$ and $\chi(\rho, \sigma)$ be two coordinate systems, η and σ being zero at the wall; and let the flow fields be represented respectively by the stream functions ψ_ζ and ψ_χ. Theorem 1. If $\bar\psi_\zeta = f(\xi, \bar\eta)$ is given, then the boundary-layer solution with respect to χ can be found directly by the substitution formula $\bar\psi_\chi = f(\xi_\chi, \bar\eta_\chi)$ where $\xi_\chi = \xi(\rho, 0)$, $\bar\eta_\chi = \bar\sigma(\partial\eta/\partial\sigma)_{\sigma=0}$. Here the bar indicates that the variable is normalized by the square root of the kinematic viscosity coefficient. Theorem 2. The coordinate system $\zeta(\xi, \eta)$ is a particular optimal system, in the sense that, as $\bar\eta \to \infty$, $\bar\psi_\zeta(\xi, \bar\eta) \to u_{e\omega}\bar\eta$ and $\psi_\zeta'(\xi, \bar\eta) \to \psi'_{e\omega}$, $u_{e\omega}$ and $\psi'_{e\omega}$ being respectively the external velocity and the external stream function due to displacement thickness at $\eta = 0$; if $\xi = \psi_e'$, $\eta = \psi_e$ (the subscript e again indicates external potential field). Any other system $\chi(\rho, \sigma)$ is optimal if and only if it is related to the above system by $\rho = f_1(\xi)$, $\sigma = f_2(\xi)\eta$, f_1 and f_2 being arbitrary functions. The flow field given by the boundary-layer approximation is the same for all optimal systems but will be different if any other system is chosen.

It might be pointed out that the choice of $\eta = \psi_e$ eliminates a large class of problems where separation occurs. Difficulty will also be expected to arise if the external field is rotational. In that case the limiting process defined above has to be re-examined. *Y. H. Kuo* (Ithaca, N. Y.).

Morris, D. N., und Smith, J. W. Ein Näherungsverfahren für die Integration der laminaren, kompressiblen Grenzschichtgleichungen. Z. Angew. Math. Mech. **34**, 193–194 (1954).

Lighthill, M. J. The response of laminar skin friction and heat transfer to fluctuations in the stream velocity. Proc. Roy. Soc. London. Ser. A. **224**, 1–23 (1954).

The laminar boundary layer in two-dimensional flow about a cylindrical body, when the velocity of the oncoming flow relative to the body oscillates in magnitude but not in direction, is analyzed mathematically. It is found that the maxima of skin friction at any point anticipate the maxima of the stream velocity, because the pressure gradient needed to speed up the main stream locally produces a given percentage increase in the slow flow near the wall sooner than it can do so in the main stream itself. For each point on the body surface there is a critical frequency ω_0, such that for frequencies $\omega > \omega_0$ the oscillations are to a close approximation ordinary 'shear waves' unaffected by the mean flow; the phase advance in the skin friction is then 45°. For frequencies $\omega < \omega_0$ on the other hand, the oscillations are closely approximated as the sum of parts proportional to the instantaneous velocity and acceleration of the oncoming stream; the phase advance in the skin friction is then $\tan^{-1}(\omega/\omega_0)$. The part depending on the instantaneous velocity may be called the quasi-steady part of the oscillations. The coefficient of the acceleration of the oncoming stream in the frictional drag of the body may be called the frictional component of the virtual mass. For a flat plate in a stream of speed V, $\omega_0 = 0.6V/x$ at a distance x from the leading edge. If c is the length of the plate, its transient motion parallel to itself is governed solely by quasi-steady forces and this added virtual mass provided that $\omega c/V < 0.6$. The frictional component of the virtual mass of a flat plate or any thin obstacle is found to be approximately 0.5 times the mass of the fluid in the boundary layer's 'displacement area'; it is suggested that the coefficient may need to be increased to about 0.8 for turbulent layers.

When the body surface is hot, the maxima in heat transfer from it tend to lag behind those of the stream velocity, as a result of thermal inertia, but this is counteracted to some extent by the effect of convection by the phase-advanced

velocities near the wall. For layers with a favourable gradient in the mean flow, one finds that the tendency to lag predominates. For the Blasius layer, however, the two effects appear to cancel out fairly closely; and for layers with adverse pressure gradient in the main stream there seems to be phase advance at the lower frequencies. At frequencies well above ω_0 there is always a phase lag of 90°, but the amplitude of heat-transfer fluctuations is then much reduced, even though that of the skin friction fluctuations is increased.

Special attention is paid to the phase lag in the heat transfer from a heated circular wire in a fluctuating stream, in the range of Reynolds number for which a laminar boundary layer exists. Curves for the amplitude and phase of the heat-transfer fluctuations as a function of frequency are given, from calculations for the layer of nearly uniform thickness which covers the front quadrant of the wire and across which most of the fluctuating part of the heat transfer is believed to occur. For frequencies small compared with $\omega_0=20V/d$ (where d is the diameter), the departure of the heat-transfer fluctuations from their quasi-steady form consists essentially of a time lag of the order of $0.2d/V$. (Author's summary.) *Y. H. Kuo* (Ithaca, N. Y.).

Illingworth, C. R. The effect of heat transfer on the separation of a compressible laminar boundary layer. Quart. J. Mech. Appl. Math. 7, 8–34 (1954).

The author develops an approximate method for dealing with a compressible laminar boundary layer with non-uniform free-stream velocity and wall temperature, and applies it to investigate the problem mentioned in the title. The approximations are made by means of a method used by Lighthill in his discussion of heat transfer in boundary layers [Proc. Roy. Soc. London. Ser. A. **202**, 359–377 (1950); these Rev. **12**, 218].

All examples worked out are for the case of uniform temperature. The effect of pressure gradient is found to vary with the ratio of stream temperature to wall temperature, but shows very little independent variation with the free-stream Mach number. This is explained in terms of the fact that the effects of pressure gradient are due mainly to its action on the fluid very near to the wall, and so are changed if and only if the density (and the viscosity) of the fluid are changed there. *C. C. Lin* (Ithaca, N. Y.).

Monaghan, R. J. An approximate solution of the compressible laminar boundary layer on a flat plate. Ministry of Supply [London], Aeronaut. Res. Council. Rep. and Memoranda no. **2760** (1949), 24 pp. (1953).

The author obtains simple formulae, based on certain simplifying approximations, for the calculation of the displacement thickness and the velocity distribution of the compressible laminar boundary layer on a flat plate with variable wall temperature. The results calculated from these formulae are in very close agreement, at least up to $M_1=5.0$, with some representative cases obtained by numerical integration of more exact formulae. The major assumption, which forms the basis of this method of approximation, is that enthalpy and velocity are dependent only on local conditions. The enthalpy-velocity relation thus derived is found to be a close approximation to Crocco's results [Consiglio Naz. Ricerche. Publ. Ist. Appl. Calcolo no. **187** (1947); these Rev. **10**, 75] for two values of the Prandtl number at least up to 80% of the free-stream velocity.
C. C. Lin (Ithaca, N. Y.).

Spalding, D. B. Mass transfer in laminar flow. Proc. Roy. Soc. London. Ser. A. **221**, 78–99 (1954).

The author applies the integral method to calculate mass transfer in the boundary layer. The chief difference between the problems of mass and heat transfer lies in the boundary condition, which now depends on the rate of transfer. Expressions are derived for the rate of mass transfer from (a) a flat plate in a longitudinal fluid stream, (b) a vertical flat plate by natural convection, (c) the forward stagnation point of a sphere in a fluid stream. Only outward transfer is considered. These calculations are particularly relevant to the combustion of liquid fuels. *C. C. Lin.*

Spalding, D. B. Mass transfer from a laminar stream to a flat plate. Proc. Roy. Soc. London. Ser. A. **221**, 100–104 (1954).

The approach of the paper reviewed above is now used to calculate the mass-transfer rate from a laminar stream to a flat plate for fluids for which the diffusion coefficient is not greatly different from the kinematic viscosity. Particular attention is paid to the very high rates of transfer. In such cases, the approximation of the velocity and concentration profiles have to be handled with greater care than in the cases of outward transfer treated in the paper reviewed above. *C. C. Lin* (Ithaca, N. Y.).

Bass, J. Space and time correlations in a turbulent fluid. I. Univ. California Publ. Statist. **2**, 55–83 (1954).

The essentially new feature in this paper is the consideration of a turbulent field which is an idealization of that behind a grid in a wind tunnel. It is stationary in time, homogeneous and isotropic in every transverse section, but not homogeneous in the longitudinal direction. This type of field is referred to as cylindrical turbulence. The author raises some questions about possible contradictions between theory and the usual interpretation of the experimental results.
C. C. Lin (Ithaca, N. Y.).

Batchelor, G. K., and Proudman, Ian. The effect of rapid distortion of a fluid in turbulent motion. Quart. J. Mech. Appl. Math. **7**, 83–103 (1954).

This paper is concerned with the calculation of the changes produced in a homogeneous turbulent motion when the fluid is subjected to a superimposed uniform distortion. The distortion is assumed to occur so rapidly that the contribution to the change in relative position of fluid particles from the turbulence is negligible. On this basis G. I. Taylor [Z. Angew. Math. Mech. **15**, 91–96 (1935)] found the effect of an arbitrary distortion on a purely sinusoidal velocity field, and in this paper the effect on homogeneous turbulence is obtained by integrating over all such Fourier components. In the case of turbulence which is initially isotropic, the relative changes in the energies of the three velocity components are found to be independent of the properties of the turbulence and are determined numerically.

Asymptotic results for the important practical case in which one of the three extension ratios is large compared with unity are presented. For the case of a large symmetrical contraction (the common wind-tunnel case), of area ratio c^{-1}, the effect on any one Fourier component depends on c in a manner which tends asymptotically to the laws given by Prandtl, but this is not so for the component energies of the whole turbulent motion since the energy after contraction is dominated by a decreasingly small range of wave-numbers. The same asymptotic dependence on c is

found for all kinds of homogeneous turbulence, although the numerical constants vary. When one extension ratio of an otherwise arbitrary distortion becomes large compared with the other two, the asymptotic effect on isotropic turbulence is the same, in all important respects, as the effect of a large symmetrical contraction, the reason being that the vorticity ultimately is everywhere parallel to the line of greatest extension.

No comparison with experiment is possible, since it can be shown that in none of the cases in which measurements have been made is the distortion sufficiently rapid for the above linear theory to be valid. Some of the distortions occurring in aerodynamic practice are rapid enough for the theory to apply, but these will be uncommon, and a consideration of the non-linear effects is very desirable. (From the author's summary.)

The authors also stated that this work was contained in a thesis of the second of them, submitted in 1951, and that it goes rather further than the work by Ribner and Tucker [NACA Tech. Note no. **2606** (1952)], which contains methods and results similar to those in the present paper.

C. C. Lin (Ithaca, N. Y.).

Rotta, J. C. Similarity theory of isotropic turbulence. J. Aeronaut. Sci. **20**, 769–778, 800 (1953).

The author considers a three-parameter family of spectrum curves instead of a four-parameter family suggested by von Kármán and Lin [Rev. Modern Physics **21**, 516–519 (1949); these Rev. **11**, 226]. Due to this restriction, he obtains only two of the three theoretically known laws of decay; namely, $u'^2 \propto t^{-5/2}$ in the limit $R\to 0$, and $u'^2\propto t^{-10/7}$ in the limit $R\to\infty$, but not the law $u'^2\propto t^{-1}$. This last relation, however, has been found to be in good agreement with experiments for the early part of this decay process. The author objects to its derivation from similarity considerations, pointing to the fact that the spectral law $F(k)\sim k$ for low wave numbers has not been found experimentally. (It should be pointed out that this particular spectrum is not a necessary condition for the validity of the particular law of decay.) The author also discusses experimental data showing the change-over from the law $\lambda^2=10\nu(t-t_0)$ to $\lambda^2=7\nu(t-t_0')$, but did not mention that this has been previously predicted theoretically in the reference cited above.

C. C. Lin (Ithaca, N. Y.).

Tchen, Chan-Mou. Transport processes as foundations of the Heisenberg and Obukhoff theories of turbulence. Physical Rev. (2) **93**, 4–14 (1954).

The author investigates the transfer of energy among the various harmonic components of motion in a viscous fluid. It is known that one principal difficulty in this basic treatment is the evaluation of the phase correlation. This the author treats by statistical considerations. Such a treatment enables the author to derive both the Heisenberg and the Obukhoff transfer functions. It is concluded that both theories are reasonable even though the Obukhoff theory is essentially inadequate for small scales. The same type of analysis is applied to shear flow, and the spectral laws for energy and shear are derived and compared with measurements. In the non-viscous range, the spectral laws of energy are $F\sim k^{-5/3}$, k^{-1}, and the spectral laws of shear are $F\sim k^{-7/3}$ k^{-1}, respectively for small and large shear. *C. C. Lin.*

Nikol'skiĭ, A. A. Problems of gas flow at sonic speed. Doklady Akad. Nauk SSSR (N.S.) **94**, 401–404 (1954). (Russian)

Consider a steady plane potential flow containing a straight sonic line A_1A_2, crossed at right angles by the fluid, with expansions at both ends which deflect the flow in opposite directions through an angle β_1. In the symmetrical supersonic region bounded by A_1A_2 and the final characteristics A_1C_1 and A_2C_1 of the expansion fans the author seeks solutions of Chaplygin's equations (1) $\partial\varphi/\partial\beta=P(\tau)\partial\psi/\partial\tau$, $\partial\varphi/\partial\tau=Q(\tau)\partial\psi/\partial\beta$ for the velocity potential and stream functions φ and ψ with boundary conditions (2) $\varphi=0$, $\psi=-\psi_1(+\psi_1)$ on $A_1C_1(A_2C_1)$, where β is the angle between the velocity vector and the axis of symmetry, τ is the square of the ratio of local to maximum speed, and $P(\tau)$ and $Q(\tau)$ are known functions. Let $L_1(L_2)$ be the line through $A_1(A_2)$ parallel to the final velocities at those points. Reflect $A_1A_2C_1$ repeatedly in L_1 and L_2 and their successive images, and require φ and ψ to be continuous at common points of two adjoining regions. By this "analytic continuation" the characteristic initial-value problem (1), (2) is reformulated with initial data (3) $\varphi=0$, $\psi=(2k-1)\psi_1$ for $2k-1\leq\psi/\psi_1\leq 2k+1$, $k=0, \pm1, \cdots$, on the image $\tau=\tau_0$ of the sonic line. By superposing solutions $\psi_\nu=z_\nu\exp(\pm2\nu\beta)$, $\varphi_\nu=(-i/2\nu)z_\nu'\exp(\pm2\nu\beta)$, where $\nu=\pi n/2\beta_1$, and using two independent solutions of $[P(\tau)z']'+4\nu^2Q(\tau)z=0$ for each ν, φ and ψ are expanded in Fourier series satisfying (3) and (2) which even serve for the entire flow flaring out of a sonic duct of width A_1A_2 into the truncated wedge-shaped region bounded by L_1 and L_2. *J. H. Giese.*

Sauer, R. Hyperbolische Probleme der Gasdynamik mit mehr als zwei unabhängigen Veränderlichen. Z. Angew. Math. Mech. **33**, 331–336 (1953). (English, French and Russian summaries)

A survey of mathematical methods in hyperbolic problems of gas dynamics in more than two variables, with special emphasis on the linearized theory. *D. Gilbarg.*

Sakurai, Akira. On the propagation and structure of a blast wave. I, II. J. Phys. Soc. Japan **8**, 662–669 (1953); **9**, 256–266 (1954).

A certain literature, most of it rather inaccessible [but see R. H. Cole, Underwater explosions, Princeton, 1948], exists on the subject of the spherical blast wave, generated when a finite amount of energy is released at a point in a homogeneous gas, during the period when the shock wave forming the front of the blast wave is still very strong. In this literature the solution obtained by G. I. Taylor during the last war, and later published [Proc. Roy. Soc. London. Ser. A. **201**, 159–174 (1950)], represents the first approximation. It is a similarity solution valid only in the earliest stages of the explosion. Similar problems exist if the energy is released uniformly along a line (leading to a cylindrical wave) or along a plane (leading to two plane waves).

The treatment of the subject in the two papers under review is the best which the reviewer has seen. The solution is carried to a second approximation in all three cases of plane, cylindrical and spherical waves, and very full numerical details indeed are obtained for a gas whose ratio of specific heats γ is 1.4. (The first approximation is evaluated also for other values of γ.) For the spherical wave, with $\gamma=1.4$, the pressure behind the shock wave at distance R from a point where a quantity of energy E was released is $0.156ER^{-3}+2.07p_0$ to a second approximation, where p_0 is

the pressure of the undisturbed gas. In the cylindrical and plane cases it is $0.212ER^{-2}+2.16p_0$ and $0.345ER^{-1}+2.33p_0$, respectively, where E is the energy released per unit length, or unit area, respectively. These formulae and other related ones are compared, favourably, with the rather limited experimental results at the author's disposal.

The procedure used by the author to obtain these results was to use as independent variables $x=r/R$ and $y=(C/U)$ instead of r and t. Here $U=dR/dt$ is the shock velocity and C is the speed of sound in the undisturbed gas, so that y is a function of t alone, but x a function of r and t. When the ratios of the fluid velocity, pressure and density to their undisturbed values are expressed as functions of x and y, the Rankine-Hugoniot equations at the shock wave become a rather simple boundary condition at $x=1$. Instead of a condition at $x=0$ one has the condition of constant total energy E. The equations of motion become rather more complicated, but the author shows that they can be solved by power series in y. The successive coefficients satisfy ordinary differential equations in x, which have to be solved numerically under the boundary conditions mentioned above. *M. J. Lighthill* (Manchester).

Fujikawa, Hiroomi. The lift on the symmetrical Joukowski aerofoil in a stream bounded by a plane wall. J. Phys. Soc. Japan 9, 233–239 (1954).

Fujikawa, Hiroomi. Note on the lift acting on a circular-arc aerofoil in a stream bounded by a plane wall. J. Phys. Soc. Japan 9, 240–243 (1954).

In these two papers the method of Green [Quart. J. Math., Oxford Ser. 18, 167–177 (1947); these Rev. 9, 113] for the case of the two-dimensional aerofoil in a bounded stream is used to determine the lift on the aerofoil in the form of a power series. Green's results are extended slightly by calculating further coefficients in this power series. The results are then applied to the special cases of a symmetrical Joukowski aerofoil of small thickness, and a circular arc aerofoil, and it is shown that for small angles of attack as used in practice, the wall effect on the lift in both cases is quite similar to that in the case of a plane aerofoil. The numerical effect of thickness in the one case and camber in the other is also noted. *R. M. Morris* (Cardiff).

Krasil'ščikova, E. A. Unsteady motion of a profile in a compressible fluid. Doklady Akad. Nauk SSSR (N.S.) 94, 397–400 (1954). (Russian)

This paper considers linearized flow about a thin airfoil which moves in an xz-plane with its chord approximately on the x-axis and starts from rest at time t_0. The potential function is given by

$$\pi\varphi(t,x,z)=-\iint_S \varphi_z(\tau,\xi,0) \times[(t-\tau)^2-(x-\xi)^2/a^2-z^2/a^2]^{-\frac{1}{2}}d\tau d\xi,$$

where a is the undisturbed speed of sound and S is the intersection of $z=0$ and the nappe extending into the past of the sound cone with vertex (t,x,z). At points of the xt-plane unaffected by the motion, the normal velocity $\varphi_z=0$. At points of $z=0$ traversed by the chord φ_z can be determined from knowledge of the airfoil's motion, mean camber, etc. The remainder of $z=0$ is divided into strips by the sound waves emitted from the ends of the chord at $t=t_0$ and their successive reflections from the paths of these ends. Integral equations for φ_z in these regions are obtained by imposing $\varphi=0$ ahead of the profile and $\varphi_t=0$ behind. $\varphi_z(t,x,0)$ is obtained by solving these equations, strip by strip, as in the author's similar treatment of steady linearized potential flow over thin finite wings [Moskov. Gos. Univ. Učenye Zapiski 154, Mehanika 4, 181–239 (1951); these Rev. 14, 815]. *J. H. Giese.*

Šneerson, B. L. Some problems on the motion of viscous fluids applied to geology. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. 1953, 500–513 (1953). (Russian)

L'auteur étudie d'abord le problème aux limites suivant. Dans le plan Oxy (Oy étant la verticale ascendante), considérons les domaines: S_1 tel que $x^2+y^2\leqq R^2$, $y\geqq 0$; et S_2 tel que: $x^2+y^2\geqq R^2$, $x^2+y^2\geqq R^2$, $0\leqq y\leqq h$, $h>R$. A l'instant initial S_i ($i=1, 2$) est rempli avec des liquides visqueux dont les densités et les coefficients de viscosité valent respectivement ρ_i et μ_i. Dans le mouvement des liquides qui se produit, on suppose la continuité des vitesses et des tensions à travers la surface de séparation; pour $y=0$ et $y=h$, la vitesse est horizontale. On obtient un problème voisin en remplaçant sur $y=h$ la condition ci-dessus par celle d'annulation des tensions normales. Les forces d'inertie sont négligées. Le problème consiste à déterminer, à l'instant initial, les vitesses dans S_1 et S_2 et principalement, le long du demi-cercle $x^2+y^2=R^2$. Dans le cas $h=\infty$, l'auteur utilise la méthode de Muskhelishvili pour former la solution sous forme de séries numériquement calculables. Il passe de là au cas général au moyen d'un artifice.

L'auteur tente une interprétation géologique du problème qu'il résout. Le processus tectonique est d'une extrême lenteur; on peut donc négliger effectivement les forces d'inertie, même le terme $\rho\partial V/\partial t$. Les équations ci-dessus donneraient les lois—à une échelle des temps convenable—de la déformation géologique des coupoles de sel. La concordance avec les faits observés ne peut être que qualitative. On lira avec intérêt les développements que l'auteur consacre à ces questions. *J. Kravtchenko* (Grenoble).

Have

★Heins, Albert E., and Feshbach, Herman. On the coupling of two half planes. Proceedings of Symposia in Applied Mathematics, Vol. V, Wave motion and vibration theory, pp. 75–87. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. $7.00.

Sound waves of complex velocity potential

$$\phi^i=\exp(ik_xx+ik_yy),$$

where k_x and k_y are constants and a time-factor $\exp(-i\omega t)$ is suppressed, are incident in the half-space $y\leqq 0$ on a wall in the plane $y=0$. If the wall were perfectly reflecting, these would be reflected waves with velocity potential $\phi^r=\exp(ik_xx-ik_yy)$. In the problem discussed here, the two half-planes $y=0$, $x\gtreqless 0$ which form the wall have different acoustic properties, each described by a different complex parameter Y, the admittance of the surface; each Y can be expressed in the form $\gamma-i\sigma$, where $\gamma(\geqq 0)$ is the conductance, σ the susceptance of the material. The total velocity potential ϕ then has to satisfy the boundary conditions

$$\frac{\partial\phi}{\partial y}=i\omega\rho Y_1\phi\ \ (y=0, x<0),\quad \frac{\partial\phi}{\partial y}=i\omega\rho Y_2\phi\ \ (y=0, x>0),$$

where ρ is the density of the medium $y<0$.

It is readily shown that

$$(*)\quad \phi(x,y)=\int_0^\infty i\omega\rho(Y_2-Y_1)G(x,y,x',0)\phi(x',0)dx' +\phi^i(x,y)+A_1\phi^r(x,y),$$

where $A_1=(k_y-\omega\rho Y_1)/(ky+\omega\rho Y_1)$. Here $G(x, y, x', y')$ is the Green's function of the differential equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+(k_x^2+k_y^2)\phi=0$$

for the half-space $y<0$ under the boundary condition $\partial G/\partial y=i\omega\rho Y_1 G$ all over the plane $y=0$. This Green's function is determined by complex-variable technique. If one puts $y=0$ in equation (*) one gets

$$\phi(x, 0)=\int_0^\infty i\omega\rho(Y_2-Y_1)G(x, 0, x', 0)\phi(x', 0)dx' +(1+A_1)\exp(ik_x x),$$

an integral of Wiener-Hopf type which is solved by the usual complex Fourier transform method. The asymptotic formula for the total field at a great distance is then found without any major difficulty. *E. T. Copson.*

Elasticity, Plasticity

Ornstein, Wilhelm. Stress functions of Maxwell and Morera. Quart. Appl. Math. 12, 198–201 (1954).

The author presents a derivation of the stress functions of Maxwell and Morera. From the derivation given in Love's "Mathematical theory of elasticity" [4th ed., Cambridge, 1927], for example, it is clear that every stress tensor satisfying the equations of equilibrium can be expressed in terms of these stress functions, a fact which the author does not establish. It is unfortunate that he did not see fit to refer to any of the papers on stress functions written in the last two decades, since some, for example Finzi's [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 19, 578–584 (1934)], are quite relevant. *J. L. Ericksen.*

Graffi, D. Über den Reziprozitätssatz in der Dynamik der elastischen Körper. Ing.-Arch. 22, 45–46 (1954).

The author has formerly [Ann. Mat. Pura Appl. (4) 18, 173–200 (1939)] obtained an elegant reciprocity theorem for transient problems in classical linear elasticity. This theorem is expressed in terms of Laplace transforms of the extrinsic and surface load functions and does not contain the unknown reactions of inertia [as does Rayleigh's extension of Betti's theorem, see Love, Theory of Elasticity, 4th ed., Cambridge, 1927, §121]. In this note the author considers the case when both displacement fields and their first derivatives vanish at $t=0$, and he supposes also that the loads acting on each system depend on time only through a common factor of proportionality. In this special case, he shows that his result reduces to Betti's theorem. He applies it to obtain a generalization of a theorem of Maxwell on the effect of a concentrated load. Finally the author shows that his original reciprocal theorem, in the case when both displacement fields and their first derivatives vanish initially, holds unchanged for a body endowed with linear viscous damping and steadily accumulative elasticity.
C. Truesdell (Bloomington, Ind.).

Woinowsky-Krieger, S. Über die Biegung von Platten durch Einzellasten mit rechteckiger Aufstandsfläche. Ing.-Arch. 21, 331–338 (1953).

In thin-plate bending theory, bending moments become infinite logarithmically in the neighborhood of concentrated lateral forces. To obtain useful information about local moments and stresses without exceeding the limitations of the theory, the concentrated force is considered as the resultant of some lateral pressure distributed over an area small compared with the least plate dimension. Nadai [Die elastischen Platten, Springer, Berlin, 1925, pp. 62 ff.] developed this method using the uniform pressure over a small circle and calculating moments in the center. The present paper extends the analysis to rectangular areas of small extent which are said to have possible physical representations as machine foundations or tire footprints. The fundamental solution is derived by integration of the Green's Function for the circular plate, and, by superposition, solutions are obtained for plates in the shape of a circle, rectangle, infinite strip and equilateral triangle.
W. Nachbar (Seattle, Wash.).

Glatzel, E., und Schlechtweg, H. Zum Problem des ebenen Spannungszustandes im kreiszylindrischen spröden Rohr unter konstantem Innen- und Aussendruck. Z. Angew. Math. Mech. 34, 81–104 (1954). (Russian summary)

Using the nonlinear stress-strain relations proposed by Schlechtweg [same Z. 14, 1–12 (1934)] for brittle materials, the authors obtain approximate solutions for a cylindrical tube subject to internal and external pressures. Estimates of errors introduced by the approximations made are given.
J. L. Ericksen (Washington, D. C.).

***Lee, E. H. Wave propagation in helical compression springs.** Proceedings of Symposia in Applied Mathematics, Vol. V, Wave motion and vibration theory, pp. 123–136. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. \$7.00.

The author is concerned with the effect of coil closure. Using simple spring theory (each element satisfies the force-longitudinal strain relationship of the whole spring), he treats the two cases of inelastic and perfectly elastic contact between coils. In the latter case, when constant velocity is given to one end, force doublets are propagated along the spring. *D. R. Bland* (Newcastle-on-Tyne).

Newlands, Margery. Lamb's problem with internal dissipation. I. J. Acoust. Soc. Amer. 26, 434–448 (1954).

The author undertakes to extend the analysis of Lamb [Philos. Trans. Roy. Soc. London. Ser. A. 203, 1–42 (1903)] for waves generated by a vertical impulse applied to the free surface of a semi-infinite frictionless solid to the case of internal dissipation. The paper is highly technical and presupposes such a broad background of mathematics, physics and seismology as to require intensive study on the part of the reader. Many of the assumptions made and equations used as starting points in the argument are based on analogies with previous work without proof of their validity. Nevertheless the analogies are clearly stated. The appendix is very confusing because of the hiatus between Fig. 1 and the discussion of LC circuitry equations. The only clue to the meaning of the author seems to be given in the brief description of the apparatus to be used in experiment given in the paragraph labeled "applications".
J. B. Macelwane (St. Louis, Mo.).

Slichter, L. B. Seismic interpretation theory for an elastic earth. Proc. Roy. Soc. London. Ser. A. 224, 43–63 (1954).

This paper is an outstandingly clear and straight-forward contribution to the field of theoretical seismology. No

better summary can be given than the following author's abstract.

"The seismic interpretation problem for an isotropic spherical earth is analyzed on the basis of elastic theory, under the assumption that the three independent elastic parameters are unknown continuous functions of the depth. It is shown that solutions for these functions may be obtained in the form of Taylor's series. The problem is treated for three types of symmetrical excitation conditions on the free surface: (1) a shear source of type P_r only; (2) a pressure distribution with vanishing surface shear stress; (3) an excitation consisting of pressure in combination with surface shear stress of type P_r. In each case the excitation functions are arbitrary functions of time. It is assumed that the associated components of surface displacement over the sphere are known from available observations, as functions of time. Thus, the complete information contained in seismic records is used in the proposed interpretation process, without need of selecting, identifying and assigning arrival times to specific events on the records. The two static elastic parameters may theoretically be determined from observations at a single frequency, including the frequency zero, or static case. The determination of the dynamic elastic parameter requires the use of at least two frequencies.

"Algebraic checks are obtained by comparing the general solutions with the corresponding results for two special cases in which the elastic parameters vary in a prescribed manner in the interior of the sphere. In both these cases treatment by the classical ray-path method of interpretation is excluded, because the wave velocity decreases with depth. Furthermore, the ray-path method (which is essentially a method of geometrical optics) would fail to distinguish between the two examples in any case, since the velocity function is the same in both, although the elastic parameters differ. In contrast to the valuable ray-path method, the analytical procedures in the present solution of the elastic problem are prohibitively cumbersome. Practical application of elastic theory to the direct interpretation of seismograms requires further development of the theory with probable utilization of modern high-speed computing methods." *J. B. Macelwane* (St. Louis, Mo.).

Ceban, V. G. The case of elastic-plastic collision of bars of various materials. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 200–210 (1953). (Russian)

Propagation of strain waves and the duration of impact are studied for collisions between a finite bar and a semi-infinite bar of equal and uniform cross-section. Material properties for the elastic-plastic bars are the density, Young's modulus and the initial yield strain. Three cases are examined in detail: collisions at velocities low enough to produce elastic waves only; collisions producing plastic regions in the finite bar; collisions producing plastic regions in the infinite bar. *W. Nachbar* (Seattle, Wash.).

Rubin, Robert J. Propagation of longitudinal deformation waves in a prestressed rod of material exhibiting a strain-rate effect. J. Appl. Phys. 25, 528–536 (1954).

The paper deals with the propagation of longitudinal plastic waves in a semi-infinite rod composed of a material exhibiting a strain-rate effect. If the rod is prestressed above the yield point, and if the impact stress is assumed small so that the equations of motion can be linearized, integral representations of the solutions can be obtained by the method of Laplace transforms. The asymptotic behavior of these solutions for large time and distance (from the impact point) is then found using the method of steepest descents.

The author finds that if a constant force is applied to the end of the rod, or if the end of the rod is moved at a constant velocity, there is a region of relatively large strain separated from a region of small plastic strain by a transition region. This region propagates itself with the same velocity as found previously by von Kármán and Duwez [J. Appl. Mech. 21, 987–994 (1950); these Rev. 12, 563] and Taylor [British Official Report **RC 329** (1942)] in theories of wave propagation which neglected strain-rate effects.

It is also found that the strain at the end of the rod asymptotically approaches the value predicted by von Kármán's theory. The author outlines a proof showing that this result also holds for a more complicated case investigated by Malvern [J. Appl. Mech. 18, 203–208 (1951); these Rev. 12, 882]. It should be noted that in the investigation of the above mentioned cases the author implicitly assumes that unloading does not take place. In the reviewer's opinion this is not obvious.

In the last part of the paper, the author considers the case in which impact on a semi-infinite rod occurs, but at the time $t=t_0$ the load at the impact end is suddenly reduced to zero. It is shown that the resulting unloading shock wave will be absorbed except for the case of a material which does not work-harden. In this case the complete solution of the unloading problem is obtained. *E. T. Onat.*

Stroh, A. N. The formation of cracks as a result of plastic flow. Proc. Roy. Soc. London. Ser. A. 223, 404–414 (1954).

Nach der Auffassung von Griffith wird der spröde Bruch von amorphen Stoffen durch schon früher vorhandene Sprünge verursacht, bei Materialien vom Typ der Metalle ist dagegen solch eine Erklärung unwahrscheinlich und Mott hat deshalb angenommen, dass die Konzentration der Spannungen um eine Häufung von Verwerfungen die Ursache der Entstehung eines Bruches ist. Der vorliegende Artikel ist eine theoretische Ausarbeitung dieses Problems. Das mathematische Problem ist das Auffinden von Lösungen für die Spannungskomponenten, die im Unendlichen verschwinden, an der Oberfläche des Sprunges vorgeschriebenen Bedingungen genügen und selbstverständlich die Gleichungen des elastischen Gleichgewichtes befriedigen. Legt man die x-Achse senkrecht zur Sprungrichtung und die y-Achse in dieser Richtung, so erhält der Verfasser z.B. für den Fall, dass die Gleitrichtung senkrecht zur Richtung des Sprunges steht

$$p_{xx}=-D(r_1r_2)^{-1/2}\sin\tfrac{1}{2}(\chi_1+\chi_2)+Dr^2(r_1r_2)^{-3/2}\cos\chi\sin(\chi-\tfrac{3}{2}\chi_1-\tfrac{3}{2}\chi_2),$$

$$p_{yy}=-D(r_1r_2)^{-1/2}\sin\tfrac{1}{2}(\chi_1+\chi_2)-Dr^2(r_1r_2)^{-3/2}\cos\chi\sin(\chi-\tfrac{3}{2}\chi_1-\tfrac{3}{2}\chi_2),$$

$$p_{xy}=Dr^2(r_1r_2)^{-3/2}\cos\chi\cos(\chi-\tfrac{3}{2}\chi_1-\tfrac{3}{2}\chi_2).$$

r, r_1 und r_2 bedeuten hier die Entfernungen des fraglichen Punktes, in dem wir den Spannungszustand untersuchen, von der Mitte, vom Anfang und vom Ende des Sprunges und χ, χ_1 und χ_2 sind die Winkel welche diese Verbindungsgeraden mit der zum Sprunge senkrechten Richtung einschliessen. D ist eine Konstante, die sich aus den auf die Verwerfungen beziehenden numerischen Daten ergibt. Zur Herleitung des Resultates benützt der Verfasser die von I. W. Busbridge [Proc. London Math. Soc. (2) 44, 115–129 (1938)] herrührende Untersuchung bezüglich eines Systems von Integralgleichungen. *T. Neugebauer* (Budapest).

MATHEMATICAL PHYSICS

Poincelot, Paul. Sur la vitesse de groupe. C. R. Acad. Sci. Paris **238**, 1289–1291 (1954).

This paper adds little to an earlier paper [same C. R. **234**, 599–602 (1952); reported by title only in these Rev. **13**, 599], both papers dealing with the propagation of disturbances in a dispersive medium of the type familiar in wave mechanics with a phase velocity c/n where c is the velocity of light and $n=(1-a^2/\omega^2)$, $a=\text{const.}$, $\omega=\text{frequency}$. The underlying question of interest is this: How fast do signals travel in such a medium? On a disturbance $f(x, t)$ in the domain $-\infty<t<\infty$, $0\leq x<\infty$ the author imposes the boundary condition $f(0, t)=0$ for $t<0$, $f(0, t)=\exp(i\Omega t)$ for $t>0$, and writes as the appropriate expression for the disturbance

$$f(x, t)=\frac{1}{2\pi i}\int_{C_1}\frac{\exp(i\omega t-\tau(a^2-\omega^2)^{1/2})}{\omega-\Omega}d\omega \quad (\tau=x/c),$$

where C_1 is an infinite straight line parallel to, and below, the real axis in the complex ω-plane. Then $f(x, t)=0$ for $t<\tau$, $f(x, t)=k(x, t)$ for $\tau<t<\theta$, and $f(x, t)=F(x, t)+k(x, t)$ for $t>\theta$, where $\theta=\tau/n$ (the time of propagation of the group), $F(x, t)$ is the permanent regime, and

$$k(x, t)=\frac{1}{2\pi i}\int_{-\pi}^{\pi}\exp(ia(t^2-\tau^2)^{1/2}\sin z)\frac{\cos(z-i\delta)}{\sin(z-i\delta)-b}dz,$$

$b=\Omega/a>1$, $\tanh\delta=\tau/t$. It follows that the front of the disturbance advances with velocity c, and behind this front there is a non-vanishing disturbance, so that the signal-velocity appears to be c. However, the precursor signals disappear rapidly in propagation and have an aperiodic character very different from the permanent regime which advances rigorously with the group velocity. In the later paper the author shows that, at infinite distance, the permanent regime, when it arrives, is established instantaneously.

J. L. Synge (Dublin).

Optics, Electromagnetic Theory

Dossier, Brigitte. Recherches sur l'apodisation des images optiques. I. Rev. Optique **33**, 2–111 (1954).

The quality of an optical image can often be improved by artificial re-distribution of the light intensity in the diffraction pattern. This can be achieved, for example, by evaporating thin films of suitable dielectric or metallic substance on to one or more surfaces of the system, or by means of specially constructed filters.

A useful application of this technique, studied by the author, is the so-called "apodization", i.e. the reduction of the intensity in the bright subsidiary maxima of the diffraction pattern. This procedure leads to improved resolution in image-forming instruments, whilst in spectroscopes it facilitates the study of structures with components of very low intensity. Various methods of apodization are discussed and it is suggested, that the modification of the phase alone is not adequate for some purposes; modifications of both amplitude and phase are in general necessary. The types of apodization screens, which for a given degree of apodization give the greatest transmission are studied. The theoretical discussion is supplemented by some experimental results.

E. Wolf (Manchester).

Dossier, Brigitte. Recherches sur l'apodisation des images optiques. II. Rev. Optique **33**, 147–178 (1954).

It is shown that in the case of rectangular apodization screens it is convenient to expand the pupil function into a Fourier series, whilst in the case of circular screens expansions in terms of Dini-Bessel functions seem useful. Expressions are given for the disturbance in the diffraction pattern in terms of the coefficients of the pupil function. Various parameters relating to the pupil function and the diffraction pattern are summarized in a tabulated form.

E. Wolf.

Storer, J. E., and Sevick, J. General theory of plane-wave scattering from finite, conducting obstacles with application to the two-antenna problems. J. Appl. Phys. **25**, 369–376 (1954).

The variational techniques of Levine and Schwinger are used to obtain an approximate solution for the free-space scattering by a number of finite, perfectly conducting obstacles in which the mutual interactions or coupling are taken into account. The far-zone scattered field is found in terms of the obstacle currents. Variational principles for the total and backscattering cross-sections are included. The case of two parallel wires of finite length is worked out and the results are checked against experiments.

C. J. Bouwkamp (Eindhoven).

Bekefi, G. The impedance of an antenna above a circular ground plate laid upon a plane earth. Canadian J. Physics **32**, 205–222 (1954).

The author is concerned with the problem of calculating the impedance of a thin wire antenna erected at the center of a circular ground plate resting upon a flat conducting earth. An integral equation is developed for the radial electric field at the surface of the earth. The expression for the impedance is thrown into a variational form. By a suitable choice of trial function for the radial electric field, approximations to the impedance are found. Special attention is paid to the case of a large ground plate. Earlier results of various authors are shown to be included in the author's. The analysis involves typical integrals of Bessel functions, among others $\int_0^\infty H_0^{(1)}(ay)H_0^{(1)}(by)dy$, which is evaluated in the appendix.

C. J. Bouwkamp.

Durand, Emile. Solution des équations de Maxwell et des équations de Dirac pour des conditions initiales données. J. Phys. Radium (8) **15**, 281–287 (1954).

Verfasser beweist zunächst die in den räumlichen Variablen x_1, x_2, x_3 und der Zeit t bestehende Identität

$$4\pi\psi(x_i, ct)=\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\iint_{(\Omega)}d\Omega\int_1^{ct}\psi(x_i+rn_i, ct-r)rdr, \quad i=1, 2, 3;$$

$$d\Omega=r^2\sin^2\vartheta d\varphi,\quad 0<\vartheta<\pi,\quad 0<\varphi<2\pi,$$

$$n_1=\sin\vartheta\cos\varphi,\quad n_2=\sin\vartheta\sin\varphi,\quad n_3=\cos\vartheta.$$

Der Integrationsbereich Ω ist durch eine Kugel vom Mittelpunkt x_i und Radius ct gegeben ($c=\text{Konstante}$). Mit Hilfe dieser Identität lassen sich Lösungen der Maxwellschen Feldgleichungen angeben, welche zur Zeit $t=0$ gegebene Anfangsbedingungen erfüllen. Als Spezialfall wird angenommen, dass die Anfangsbedingungen von einer oder von zwei der Raumkoordinaten unabhängig sind. Die dieser Spezialisierung entsprechende Identität wird bestimmt.

Weiterhin behandelt Verfasser Dirac's Gleichungen des Elektrons. Die hier grundlegende Identität

$$\psi(x_1, ct) = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta - k_0^2\right)\int_0^t \frac{d\tau}{c(t-\tau)}\frac{\partial}{\partial t}\int_0^{c(t-\tau)} \bar{\psi} I_0(k_0\gamma) r^2 dr$$

enthält die Besselfunktion I_0. Ferner gilt

$$\bar{\psi} = \frac{1}{4\pi}\int\int \psi(x_i + rn_i, c\tau)d\Omega, \quad \gamma = [c^2(t-\tau)^2 - r^2]^{1/2}.$$

Damit ergeben sich Lösungen der Diracschen Gleichungen für gegebene Anfangsbedingungen. Speziell werden wiederum Funktionen ψ_i ($i = 1, 2, 3, 4$) behandelt, die nicht von z oder auch solche, die nicht von y und z abhängen.

M. Pinl (Köln).

Kruskal, M., and Schwarzschild, M. Some instabilities of a completely ionized plasma. Proc. Roy. Soc. London. Ser. A. 223, 348–360 (1954).

Den Ausgangspunkt der Untersuchung bildet die Bewegungsgleichung in einem stark leitenden Plasma

$$(1) \qquad \rho\frac{d\mathbf{v}}{dt} = \mathbf{j}\times\mathbf{B} + \epsilon\mathbf{E} - \nabla p + \rho\mathbf{g},$$

wo mit ϵ die Ladungsdichte und mit $\mathbf{g}$ die Schwerebeschleunigung bezeichnet ist und alle anderen Symbole die gewohnte Bedeutung haben. Dazu kommen die Maxwellschen Differentialgleichungen und die Grundgleichungen der Hydrodynamik, die mit (1) acht Grundgleichungen liefern. Die Verfasser lösen dieses Differentialgleichungssystem in zwei Fällen. Im ersten wird das Plasma gegen die Gravitation infolge eines horizontalen magnetischen Feldes getragen, im zweiten wird das Plasma durch ringförmige magnetische Kraftlinien in einem zylinderförmigen Volumen gehalten und der in dem parallel zur Achse fliessende Strom verursacht eben das magnetische Feld. Der erste Fall ist instabil gegen eine Störung senkrecht zur Vertikalen, die eine genügend grosse Wellenlänge besitzt und der zweite gegen Deformationen des Zylinders senkrecht zur Achse, was eine bekannte physikalische Erscheinung ist. Im ersten Fall ist die strenge Lösung durch elementare Funktionen ausdrückbar, im zweiten müssen Besselsche und Hankelsche Funktionen erster Art und erster Ordnung eingeführt werden. Zum Schluss wird noch das Problem der endlichen Leitfähigkeit untersucht, weil die erwähnten Lösungen sich auf den Grenzfall der unendlichen Leitfähigkeit beziehen.

T. Neugebauer (Budapest).

Axner, Yngve. Calculation of some magnetic and electric fields with cylindrical symmetry. Appl. Sci. Research B. 4, 124–136 (1954).

(r, ϕ, z) being cylindrical coordinates, the author computes the scalar and vector potentials and the components, of an axially symmetric electromagnetic field which is also "mirror-symmetric" with respect to the plane $z=0$. All field quantities appear as power series in z whose coefficients are expressed in terms of the (presumably given) field components in the plane $z=0$. Solutions in closed form are obtained for the case that the field components in the plane $z=0$ are proportional to powers of r.

A. Erdélyi.

Moon, Parry, and Spencer, Domina Eberle. A new electrodynamics. J. Franklin Inst. 257, 369–382 (1954).

Quantum Mechanics

1953

Heisenberg, W. Zur Quantisierung nichtlinearer Gleichungen. Nachr. Akad. Wiss. Gottingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. 153, 111–127 (1953).

This paper presents a new program for the mathematical investigation of strongly non-linear equations. In Part I the program is explained by applying it to a simple and well-understood example, the quantum theory of a one-dimensional anharmonic oscillator. In Part II there is a brief discussion of the application of the method to a quantum field theory with strong interactions.

Part I. The equations of motion for the coordinate q and momentum p of an anharmonic oscillator are

$$(1) \qquad m\dot{q} = p, \quad m^{-1}\dot{p} = -\omega_0^2 q - \lambda q^3.$$

Let Φ, Φ' be any two stationary states with energies E, E'. For each pair of integers k, n there is defined a number $\tau(k|n)$ by equation

$$(2) \qquad \langle\Phi|(q(t))^k(p(t))^n|\Phi'\rangle = \tau(k|n)e^{i\omega t}, \quad \omega = E' - E.$$

The product of q and p operators in (2) is to be symmetrized before taking the matrix element. Inserting the operator equations (1) into (2) gives a set of linear equations for the $\tau(k|n)$,

$$(3) \qquad \begin{aligned} i\omega\tau(k|n) = {} & km^{-1}\tau(k-1|n+1) \\ & -m\omega_0^2 n\tau(k+1, n-1) - m\lambda n\tau(k+3|n-1) \\ & +\tfrac{1}{4}m\lambda\hbar^2 n(n-1)(n-2)\tau(k+1, n-3). \end{aligned}$$

This is an infinite set of equations to determine the eigenvalue ω. The eigenvalues are all the energy differences between pairs of stationary states of the oscillator. So far no approximations have been made.

To solve Eq. (3) in practice, some approximation must be made which has the effect of reducing the equations to a finite set. The following cut-off method is proposed. An integer N being chosen, each $\tau(k|n)$ with $k+n>N$ is expressed as a linear combination of $\tau(k|n)$ with $k+n\leqq N$ by a relation of the form

$$(4) \qquad \begin{aligned} \tau(k|n) = {} & \binom{k}{2}\Delta\tau(k-2|n) \\ & + kn\Delta\tau(k-1|n-1) + \binom{n}{2}\Gamma\tau(k|n-2) \\ & - \frac{1}{2}\binom{k}{2}\binom{k-2}{2}\Delta^2\tau(k-4|n) - \cdots, \end{aligned}$$

where

$$(5) \qquad \Delta = \tau_{00}(2|0), \quad \Lambda = \tau_{00}(1|1), \quad \Gamma = \tau_{00}(0|2)$$

are the special coefficients defined by taking both Φ and Φ' to be the ground state of the oscillator. The form of Eq. (4) is suggested by considerations borrowed from field theory. Equations (3) and (4) then give a finite set of relations between the $\tau(k|n)$ with $k+n\leqq N$.

The author constructs the equations and solves them numerically in the approximations $N=1$ and $N=3$. For weak anharmonicity ($\lambda\ll 1$) the results agree with those of an ordinary perturbation treatment. For strong anharmonicity ($\lambda\gg 1$) the $N=3$ approximation gives results differing by about 10% from the exact solutions [which were calculated by integrating the Schrödinger equation numerically].

Part II. As an example of a simple and strongly non-linear field theory, the author considers a spinor field ψ satisfying the field equation

$$(6) \qquad \gamma_\mu(\partial\psi/\partial x_\mu) + l^2\psi(\psi^+\psi) = 0$$

with l a constant of the dimension of a length. Instead of coefficients $\tau(k|n)$ we now have "Feynman amplitudes", which are defined as matrix elements of products of field operators $\psi(x)$ and $\bar{\psi}(y)$ taken at various points of space-time. Instead of Eq. (3) we have a set of integral equations connecting the Feynman amplitudes. Instead of Eq. (4) we have another set of equations expressing the Feynman amplitude with N variables in terms of amplitudes with less than N variables. Instead of the numbers Δ, Λ, Γ we have the fundamental invariant function

$$(7)\qquad S_1(x-y)=\langle 0|\psi^+(y)\psi(x)-\psi(x)\psi^+(y)|0\rangle,$$

where $|0$ represents the vacuum-state of the non-linear field.

The author's program is to solve the coupled integral equations [analogs of (3) and (4)] to determine the stationary states of the system, with a function S_1 which is guessed in advance. The form of S_1 which is chosen is simply the Green's function (7) defined for a linear spinor field, with the δ-function singularity upon the light-cone removed. Intuitive arguments are put forward for this choice of S_1. The program is to solve the equations in successively higher approximations by increasing the value of N. Since S_1 has no strong singularity, the divergences of the usual treatment will not arise. So it may be hoped that for large N the solutions will converge to give physically consistent results.

The following remarks express the reviewer's opinion of this program. In Part I the program is simply a method of approximating to the solution of a known problem. The only question is whether the method converges for large N, and this question can be answered by making further calculations. In Part II, however, an entirely new physical assumption is made by the arbitrary choice of the function S_1. Even if the approximation method converges well, there is no check on the choice of S_1, and a different choice of S_1 might lead to physical results which were equally convergent but quite different. So the program of Part II represents a radical departure from the physical content, as well as the approximation method, of orthodox field theory. G. Källèn [private communication] has shown that the use of a non-singular S_1 is definitely inconsistent with the usual rules of quantum mechanics. These questions are further discussed in the paper reviewed below. *F. J. Dyson.*

Heisenberg, W. Zur Quantentheorie nichtrenormierbarer Wellengleichungen. Z. Naturforschung **9a**, 292–303 (1954).

This paper is mainly an amplification of the second part of the paper reviewed above. Two important new points are added. In Section 1 the author explains the physical ideas which underlie his program. In Section 4 the program is applied to calculate numerically the rest-mass of the lightest spinor particle, in the non-linear theory of a spinor field ψ satisfying the field equation

$$(1)\qquad \gamma_\nu(\partial\psi/\partial x_\nu)+l^2\psi(\psi^+\psi)=0.$$

Section 1. The author divides the total Hilbert space of orthodox field-theory into two parts. Hilbert Space I is spanned by all states of the fields having a total energy less than some upper bound M_g. Hilbert Space II is spanned by states with energy greater than M_g. The bound M_g is finite but so large as to be unattainable in any real physical situation. The author assumes that the ordinary rules of quantum mechanics apply only in Hilbert Space I. Hilbert Space II does not really exist, that is to say it is considered to be a "symbolic" Hilbert space, and its only effect is to remove the singularities which occur at very short distances in the Green's Functions of orthodox field theory.

Section 4. Working to the approximation $N=3$ of his program [see preceding review], the author finds for the mass of the lightest spinor particle

$$(2)\qquad E=7.45l^{-1},$$

where l is the fundamental length of the theory. The factor 7.45 is a pure number; precisely, it is

$$(3)\qquad 2^{5/4}\pi^{3/4}[1+24\log 2-(63/4)\log 3]^{-1/4}.$$

It happens also to be close to the observed ratio of the proton to the pi-meson mass.

The reviewer feels that Hilbert Space II is a vaguely defined concept. The author's choice of the function S_1 [see preceding review] does not clearly follow from the stated properties of Hilbert Space II. To choose for S_1 a Green's function from the linear field theory seems somewhat inconsistent with the author's original intention to avoid linear approximations entirely. *F. J. Dyson.*

Stiegler, Karl Drago. Sur les rapports entre le principe de Maupertuis-Lagrange et celui de Fermat d'une part et la théorie de la relativité restreinte et la mécanique ondulatoire d'autre part. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 1052–1063 (1953).

The author writes down nine axioms which include a transformation

$$(\mathrm{T})\qquad x'=a_{11}x+a_{12}t,\quad y'=y,\quad z'=z,\quad t'=a_{21}x+a_{22}t,$$

with $a_{11}=a_{22}$, and two stationary principles

$$(\text{M-L})\qquad \delta\int v\,ds=0,\qquad (\mathrm{F})\qquad \delta\int ds/u=0$$

for the motion of a particle (sufficiently slow), v being the speed of the particle, u a phase velocity, and ds the spatial element. He is led at once to the de Broglie relationship $uv=C^2$ where C is a constant; C is then found to be an upper bound for the speed of a particle, and the transformation (T) is shown to be the Lorentz transformation in which C is the parameter usually denoted by c. [If the reviewer's concept of de Broglie waves is correct, the author's axioms are not adequate to deal with them. The (M-L) principle, assumed by the author for small speeds but actually used for all speeds, is not correct [cf. Synge, Geometrical mechanics and de Broglie waves, Cambridge, 1954, p. 81; these Rev. 15, 566]; the particle paths are not in general orthogonal to the phase waves [op. cit. pp. 30, 90]; and the relationship $uv=$const. holds only for unaccelerated motion [op. cit. pp. 90, 91].] *J. L. Synge* (Dublin).

Good, R. H., Jr. Hamiltonian mechanics of fields. Physical Rev. (2) **93**, 239–243 (1954).

This work develops further the covariant mechanics of fields of M. Born [Proc. Roy. Soc. London. Ser. A. **143**, 410–437 (1934)] and H. Weyl [Physical Rev. (2) **46**, 505–508 (1934)], in which the four space-time derivatives of each field component are regarded as four velocities and hence have associated with them four momenta. The author introduces Hamiltonian equations, Lagrange and Poisson brackets, and integrals of motion, as generalizations of the corresponding entities in particle mechanics. It is found that the only canonical transformations are point transformations. *N. Rosen* (Haifa).

van Winter, Clasine. The asymmetric rotator in quantum mechanics. Physica 20, 274–292 (1954).

For many physicists who are concerned with the asymmetric rotator this paper will be a very useful survey. First the symmetric rotator is discussed briefly, and the eigenvalues and eigenfunctions Φ_{JKM} defined, the subscripts referring to the values of the square of the total angular momentum and to the components along a principal axis of inertia and along a direction fixed in space. The asymmetric rotator has eigenfunctions Ψ_{JM} which are expressible as linear combinations of the Φ_{JKM}. The latter, however, are conveniently replaced by symmetrized eigenfunctions adapted to the four-group symmetry of the asymmetric rotator. The energy eigenvalues are discussed and selection rules for dipole radiation are derived. The definition of the phases of the eigenfunctions receives attention.

C. Strachan (Aberdeen).

Klein, Abraham. Convergence of the adiabatic nuclear potential. Physical Rev. (2) 91, 740–748 (1953).

To justify the use of the relativistic two-body equation as the exclusive basis for the discussion of nuclear forces in the non-relativistic domain (see preceding review) the author intends to investigate the following problems: the hard core, higher order potentials, and radiative corrections. In the present paper the leading term of the potential of order $4n$, n an arbitrary integer, of the pseudoscalar theory with pseudoscalar coupling is computed by a method which takes advantage of the special pair-character of this term. It appears that the coefficients increase with n as $n!(n-1)!$ (see, however, the following review). This conclusion is based on the calculation of the leading terms through eighth order in the coupling constant. According to this result, the series representing the potential would not converge for $x\lesssim 1$, where $x=\mu r$ is the nucleon separation measured in units of the meson Compton wavelength, μ^{-1}. *E. Gora.*

Klein, Abraham. Convergence of the adiabatic nuclear potential. II. Physical Rev. (2) 92, 1017–1020 (1953).

Results obtained in part I [see the preceding review] concerning the non-convergence of the series representing the adiabatic nuclear potentials are shown to be incorrect. A symmetry property for pair potentials of order $4n$ was overlooked there. Higher order pair diagrams which do not consist of single closed meson perimeters are cancelled in the adiabatic limit by the iterates of lower order diagrams. The correct series is derived; its summation confirms a result given previously by Wentzel [Helvetica Phys. Acta 15, 111–126 (1942)]. Similar results are obtained for two additional series of potentials: potentials "of order $4n$ with one pair fewer" which have as their leading term the one-pair potential of fourth order, and potentials of order $4n+2$ which are characterized by open meson-line perimeters with end points at each of the nucleon positions. Each series has the same radius of convergence which is determined by the condition $xe^x>2\alpha$ (for x see preceding review; $\alpha=(g^2/4\pi)(\mu/2M)$). With $g^2/4\pi=15$, the series converge for $x>0.85$; with $g^2/4\pi=10$, for $x>0.57$. For $x\lesssim 1$, convergence is still slow for such values of the coupling constant. However, treatments of the potential problem which take into account radiative corrections indicate that the pair coupling is strongly damped as a consequence of self-interactions. With $\alpha\sim 0.1$, that is the order of magnitude expected with such corrections, the series converge for $x>0.17$, and perturbation theory is applicable for distances as small as $x=0.5$. *E. Gora* (Providence, R. I.).

Klein, Abraham. Single-time formalisms from covariant equations. Physical Rev. (2) 94, 1052–1056 (1954).

In a previous paper [Physical Rev. (2) 90, 1101–1115 (1953); these Rev. 15, 844] the author outlined a method for obtaining a single-time equation from the relativistically covariant two-particle equation (R.E.). A simplified and more concise derivation is now given which is based on an alternative version of the R.E., constructed by means of free-particle Green's functions. The single-time wave equation is obtained by means of an iteration procedure which employs the solution of the R.E. for an instantaneous interaction as starting point. In order to make this single-time equation resemble as closely as possible the equation obtained from the Tamm-Dancoff formalism, a single-time wave function containing only positive energies is used. Possible generalizations which include negative energy components of the wave function are discussed, but not carried out in detail. *E. Gora* (Providence, R. I.).

Yappa, Yu. A. The "Schrödinger form" of relativistically invariant equations. Doklady Akad. Nauk SSSR (N.S.) 94, 817–820 (1954). (Russian)

This paper examines the conditions under which a general relativistically invariant wave-equation

$$(L^\mu(\partial/\partial x_\mu)+\kappa)\psi=0 \tag{1}$$

is equivalent to an equation of the form

$$[(\partial/\partial x_0)+F_\mu{}^\nu(\partial/\partial x_\nu)+\kappa G_\mu]\psi=0. \tag{2}$$

Eq. (2) is said to be of "Schrödinger form" since for $\mu=0$ it may be considered as a Schrödinger equation

$$(\partial/\partial x_0)\psi=H\psi \tag{3}$$

with a well-defined Hamiltonian operator H. The complete conditions for equivalence of (1) with (2) are found; they are too complicated to be stated here. A necessary (but not sufficient) condition for the existence of the Schrödinger form is the following: every solution of (1) must satisfy the simple Klein-Gordon equation

$$[(\partial^2/\partial x_\mu{}^2)-\kappa^2]\psi=0. \tag{4}$$

This condition excludes most of the wave-equations for particles of spin greater than 1, since these equations generally lead to equations of higher order than Eq. (4). *F. J. Dyson* (Princeton, N. J.).

Širokov, Yu. M. On a new class of relativistic equations for elementary particles. Doklady Akad. Nauk SSSR (N.S.) 94, 857–859 (1954). (Russian)

The author claims to have discovered a new class of relativistic wave equations, having the property that the wave-function for a particle of spin s has exactly $2s+1$ components. The claim, if valid, would be of great importance, for it would imply the possibility of describing the proton by a relativistic equation without requiring the existence of negative-energy solutions or of anti-particles.

The construction is as follows. Let S_1, S_2, S_3 be square matrices of $2s+1$ rows and columns, representing the infinitesimal generators of the 3-dimensional rotation group. Let $\Psi(p)$ be a $(2s+1)$-component function of the momentum-vector p, belonging to the same representation of the rotation-group. We extend this representation of the 3-dimensional group to a representation of the Lorentz group, by postulating that under the infinitesimal Lorentz transformation,

$$x_\mu\to x_\mu-\epsilon_{\mu\nu}x_\nu,\quad \epsilon_{\mu\nu}=-\epsilon_{\nu\mu}, \tag{1}$$

the $\Psi(p)$ transform according to

$$\Psi(p)\to\Psi(p)+\tfrac{1}{2}i\epsilon_{\mu\nu}M_{\mu\nu}\Psi(p) \tag{2}$$

with the $M_{\mu\nu}$ ($\mu, \nu=1, 2, 3, 4$) defined by

$$M_{12}=-M_{21}=-i\left[p_1\frac{\partial}{\partial p_2}-p_2\frac{\partial}{\partial p_1}\right]+S_3, \tag{3}$$

$$M_{14}=-M_{41}=-\left[E\frac{\partial}{\partial p_1}+p_1\frac{\partial}{\partial E}\right]+i\frac{p_2S_3-p_3S_2}{E+(E^2-p^2)^{1/2}},\ \text{etc.} \tag{4}$$

These definitions are consistent and relativistic, because the $M_{\mu\nu}$ satisfy the commutation relation required for 4-dimensional angular-momentum operators

$$[M_{\mu\nu}, M_{\lambda\sigma}]=i(\delta_{\mu\sigma}M_{\lambda\nu}+\delta_{\mu\lambda}M_{\nu\sigma}+\delta_{\nu\sigma}M_{\mu\lambda}+\delta_{\lambda\nu}M_{\sigma\mu}), \tag{5}$$

$$[M_{\mu\nu}, p_\lambda]=i(p_\nu\delta_{\mu\lambda}-p_\mu\delta_{\nu\lambda}). \tag{6}$$

There is a misprint of a sign in Eq. (11) of the paper, given correctly in Eq. (4) of this review. The author further requires Ψ to satisfy the simple wave equation

$$(\Box-k^2)\Psi=0. \tag{7}$$

The author's construction is quite correct, but his claim to have something new is incorrect. Let $\psi(p)$ be a 4-component Dirac spinor satisfying the Dirac equation

$$(p_\mu\gamma_\mu-ik)\psi(p)=0. \tag{8}$$

The law of transformation of ψ under the Lorentz transformation (1) is known. Also, the two "small" components of ψ can be expressed in terms of the two "large" components by using (8). Let us identify $\Psi(p)$ with the two large components of $\psi(p)$. Then the transformation law of $\Psi(p)$ can be deduced from that of $\psi(p)$, using (8) to eliminate the small components. The result is identical with (2), (3), (4), putting $S_j=\frac{1}{2}\sigma_j$ where the σ_j are the Pauli spin-matrices. A similar result holds also for the general case of spin greater than $\frac{1}{2}$. The author's $\Psi(p)$ is identical with a particular set of $(2s+1)$ linearly independent components of the usual relativistic wave-function for a particle of spin s. (For a general proof of this I am indebted to V. Bargmann.) The author's equations are therefore subject to exactly the same arguments as the usual equations, in connection with the existence of anti-particles.

Apart from the fact that he claims too much for it, the author's construction of the matrices $M_{\mu\nu}$ [Eq. (3) and (4) of this review] is simple and elegant and appears to be new.

F. J. Dyson (Princeton, N. J.).

Hellund, Emil J., and Tanaka, Katsumi. Quantized space-time. Physical Rev. (2) 94, 192–195 (1954).

The authors propose that the space-time of physical experience (q-space) is defined by 4 coordinates x_μ' ($\mu=1, 2, 3, 4$) which are not numbers but operators. Underlying the q-space is a C-space defined by 4 coordinates x_μ which are ordinary numbers. They propose that the x_μ' be related to the x_μ for a free particle by the equations

$$x_\mu'=x_\mu-l^2(\partial/\partial x_\nu)x_\nu(\partial/\partial x_\mu),$$

giving the commutation rules

$$[x_\mu', x_\nu']=l^2\{x_\mu(\partial/\partial x_\nu)-x_\nu(\partial/\partial x_\mu)\},$$

where l is a constant (fundamental length). The eigenvalues and eigenfunctions of the operators x_μ' are calculated. It is suggested that all physically observable states must be time-localized, having wave-functions which are eigenfunctions of x_4'.

F. J. Dyson (Princeton, N. J.).

Ioffe, B. L. On the divergence of a perturbation-theory series in quantum electrodynamics. Doklady Akad. Nauk SSR (N.S.) 94, 437–438 (1954). (Russian)

The divergence of perturbation-theory series may have either of two causes. Either (1) the equations of quantum electrodynamics may have no finite solutions, or (2) the equations have finite solutions but the perturbation theory gives divergent asymptotic series which determine the solutions only approximately. The author here considers the theory of a constant unquantized magnetic field H interacting with the quantized electron-positron field. In this case the equations have exact solutions which were calculated in closed analytic form by J. Schwinger [Physical Rev. (2) 82, 664–679 (1951); these Rev. 12, 889]. The expansion of the solutions in powers of H, which is the result of applying perturbation theory to this example, gives divergent series. So, in this simplified model of quantum electrodynamics, alternative (2) is the correct one.

Unfortunately the complete quantum electrodynamics with quantized electromagnetic field is a much more complicated theory, and we cannot conclude anything about the complete theory from the behavior of the simplified model.

F. J. Dyson (Princeton, N. J.).

Rayski, Jerzy. On a regular field theory. II. Quantized. Acta Phys. Polonica 13, 15–28 (1954). (Russian summary)

This paper gives the quantized version of the author's regular field theory, which was developed classically in an earlier paper [same Acta 11, 314–327 (1953); these Rev. 15, 82]. To quantize the theory, the author follows the method of J. Schwinger [Physical Rev. (2) 82, 914–927 (1951); these Rev. 13, 520]. The field operators obey the same non-local field equations as in the classical theory. However, the quantum-mechanical state of the system is defined by assigning eigenvalues to the field-operators only when the assignment is made on the initial or on the final hypersurface. On intermediate hypersurfaces the operators have no direct physical meaning and do not obey canonical commutation rules.

The physical consequences of the theory are all contained in the unitary transformation operator U which transforms the field operators on the initial hypersurfaces into the corresponding operators on the final hypersurface. It is shown how U may be calculated as a power series in the coupling constant g. It is proved that, if the form-factor of the theory satisfies certain integrability conditions, the power series for U is convergent for small g, and the theory therefore possesses regular solutions.

F. J. Dyson.

Anderson, James L. Green's functions in quantum electrodynamics. Physical Rev. (2) 94, 703–711 (1954).

This paper is concerned with the formal algebra of Green's functions and scattering matrix elements in quantum electrodynamics. By using notations representing functional differentiation, many of the Green's functions are expressed in a very concise and elegant form. For example, the one-electron Green's function of Schwinger becomes

$$G(x, x')=S^F(x-x')+\int\int dydy'S^F(x-y)$$
$$\times\{\partial\log s_{00}/\partial S^F(y-y')\}S^F(y'-x')$$

where S_{00} is the vacuum-to-vacuum element of the S-matrix. It is verified that the Schwinger Green's functions are

identical with the modified particle propagation functions which arise in the theory of the S-matrix. *F. J. Dyson.*

Cander, A. F. On the problem of many particles in quantum mechanics. Doklady Akad. Nauk SSSR (N.S.) **90**, 761–764 (1953). (Russian)

A treatment of the many-particle problem of nuclear theory is proposed which is based on a suggestion by Fok [Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **10**, 9–10, 961 (1940)] to distinguish between four groups of indices referring to protons and neutrons of positive and negative spin direction, respectively. The one-particle wave functions whose indices belong to one of these groups contribute to the expansion terms of the wave function of the total system a factor which can be written as a determinant $|\psi_i(r_j)|$. A discussion of the dependence of the kinetic and potential energy of the nucleus upon the number of protons and neutrons is based on this formalism. *E. Gora.*

Kinoshita, Toichiro, and Nambu, Yoichiro. The collective description of many-particle systems (a generalized theory of Hartree fields). Physical Rev. (2) **94**, 598–617 (1954).

Die Verfasser entwickeln eine systematische Methode zur Behandlung von Problemen, in denen eine grosse Zahl von Teilchen durch ein Bosesches Feld (elektromagnetisches Feld, Mesonenfeld) miteinander in Wechselwirkung stehen. Als Beispiel dient der Fall von Teilchen (Fermionen) und eines skalaren Feldes mit der Masse μ (der das Feld beschreibenden Partikeln). Für die Hamiltonsche Funktion eines solchen Systems folgt z.B.

$$H=\frac{1}{2m}\psi^*p^2\psi+\tfrac{1}{2}\{\pi^2+(\nabla\varphi)^2+\mu^2\varphi^2\}+g\psi^*\psi\varphi$$
$$\equiv H_{\text{par}}+H_{\text{mes}}+H_{\text{int}},$$

wo $\hbar$ (d.h. $h/2\pi$) und c als Einheiten benutzt werden, φ das skalare Feld beschreibt, π das zu φ konjugierte Moment ist und g eine Kopplungskonstante bedeutet. Die übrigen Symbole haben die in der Quantenmechanik gewohnte Bedeutung. Die bekannte Methode zur Behandlung eines solchen Problems ist, vom ungestörten Problem auszugehen und dann die Störungstheorie anzuwenden. Besteht jedoch das betrachtete System aus vielen Teilchen, so kann diese Störung recht gross werden, auch dann, wenn sie zwischen zwei Teilchen klein ist. Dieses Übel kann man beseitigen, wenn man den ausschlaggebenden Teil der Störung schon in die Näherung einbezieht, von der man ausgeht. Diese "kollektive" Beschreibung eines Problems wurde schon von Bohm und Pines bei der Behandlung der Plasmaoszillationen angewandt, und die vorliegende Arbeit ist eine Verallgemeinerung dieser Methode. Der wesentlichste Punkt ist eigentlich die Linearisierung einer nichtlinearen Wechselwirkung durch Einführung gewisser Mittelwerte.

Es wird gezeigt, dass die schon erwähnte Methode von Bohm und Pines, ausserdem die Hartree-Focksche und die Thomas-Fermische Methode als Spezialfälle in dem Verfahren der Verfasser enthalten sind. Zur Behandlung von Atomkernproblemen ist jedoch diese Methode, wegen der kurzen Reichweite der Atomkernkräfte, wenig geeignet.

T. Neugebauer (Budapest).

Günther, Marian. Relativistic configuration space formulation of the multi-electron problem. II. Physical Rev. (2) **94**, 1347–1357 (1954).

A formalism developed in a previous paper of the author [Physical Rev. (2) **88**, 1411–1421 (1952); these Rev. **14**, 707], which represented a generalization of the Dirac-Fock-Podolsky many-time theory, is now adapted to the Dirac hole theory. This is achieved by a generalization of the wave function concept: the wave function is now considered as a function of both the configurational variables x and the occupation numbers. "Configurational" and "background" electrons are distinguished; the latter are those which are represented by their configuration numbers only. This requires the use of a "mixed representation" which is intermediate between the configurational representation and the method of second quantization. The Heisenberg operators $A_\mu^{(H)}(x)$ appearing in the equations cannot be expressed in terms of the free radiation field operators $A_\mu(x)$ only, but depend also on the operators of the electronic background; $A_\mu^{(H)}(x)$ applied to the wave function induces changes of the occupation number of the background particles. An e-power expansion is used in working out the formalism. Whenever the particle interaction is expected to be strong, the configuration-space description is used. This is the case in the bound-state problem while background and free particles are represented in the plane-wave approximation. The relations between the proposed formalism and the Feynman formalism are discussed, in particular, the problem of time inversion for which there is no need in the author's theory; all the particles may travel in the same time direction. The formalism is applied to the bound-state problem for which it is expected to be particularly appropriate. Explicit methods of solving the resulting equations are not given, but a method of eliminating the virtual processes up to terms of the order e^4 is developed. *E. Gora.*

Gellikman, B. T. On the theory of strong coupling for meson fields. Doklady Akad. Nauk SSSR (N.S.) **90**, 991–994 (1953). (Russian)

In a previous paper of the author [same Doklady (N.S.) **90**, 359–362 (1953); these Rev. **15**, 379] a perturbation method for the treatment of the strong coupling problem has been developed which can be used if the wave function is separable into a larger classical and into a smaller operational part. The method is now applied to the pseudoscalar meson field in interaction with a nucleon, and to the corresponding problem of two interacting nucleons. *E. Gora.*

Gellikman, B. T. On the theory of strong coupling. Doklady Akad. Nauk SSSR (N.S.) **91**, 39–42 (1953). (Russian)

A perturbation method for the treatment of strong coupling [see the preceding review] is generalized by taking into consideration the motion of the heavy particles. The conditions of applicability of the method are discussed.

E. Gora (Providence, R. I.).

Gellikman, B. T. On polarization of the vacuum in the theory of strong coupling. Doklady Akad. Nauk SSSR (N.S.) **91**, 225–228 (1953). (Russian)

The contribution of the nucleonic vacuum to the nucleon-nucleon and meson-meson interactions in the strong coupling theory developed in previous papers [see the two preceding reviews] is studied in a qualitative way. The polarization of the vacuum gives rise to non-linear terms in both the nucleonic and the mesonic fields. Divergences are pointed out, but not investigated. The non-linear part of the mesonic field which does not contain divergencies appears to be small. This is contrary to a result of Yennie's [Physical Rev. (2) **88**, 527–536 (1952); these Rev. **14**, 520] who ob-

tained a large non-linearity using a method where the classical part of the wave function is not separated.
E. Gora (Providence, R. I.).

Bethe, H. A., and Maximon, L. C. Theory of bremsstrahlung and pair production. I. Differential cross section. Physical Rev. (2) 93, 768–784 (1954).

The differential cross-sections for bremsstrahlung and pair production are calculated, for initial and final electron energies large compared to mc^2, without using the Born approximation. The decisive step is the use in the initial and final states of wave functions which are essentially those proposed by Furry [Physical Rev. (2) 46, 391–396 (1934)]. It is shown that for all energies of the electron these agree with the exact wave functions of Darwin [Proc. Roy. Soc. London. Ser. A. 118, 654–680 (1928)] except for terms of relative order a^2/l^2 where $a=Ze^2/\hbar c$ and l is the angular momentum. The result for bremsstrahlung is found to agree with the Bethe-Heitler result except for a multiplicative factor. The same holds for pair production except that the multiplicative factor is different, and another term of similar structure appears. The error in calculated cross-sections is estimated to be of the order of l/ϵ where ϵ is the energy of the final electron in bremsstrahlung or that of the less energetic electron in pair production, in units of mc^2.
A. Salam (Cambridge, England).

Davies, Handel, Bethe, H. A., and Maximon, L. C. Theory of bremsstrahlung and pair production. II. Integral cross section for pair production. Physical Rev. (2) 93, 788–795 (1954).

The differential cross-section for bremsstrahlung and pair production calculated by Bethe and Maximon [see the preceding review] has been integrated over all angles. For small Z, the correction to the Born approximation is proportional to Z^2; for large Z it is somewhat less. The correction is shown to arise for large recoil momenta of the nucleus and is thus practically unaffected by screening, which is important only for small recoil momenta. Agreement with experiments at 88 and 280 Mev. is found to be excellent.
A. Salam (Cambridge, England).

Breit, G., and Bethe, H. A. Ingoing waves in final state of scattering problems. Physical Rev. (2) 93, 888–890 (1954).

Bethe and Maximon, in the paper reviewed second above, have made the important observation that in the matrix element for bremsstrahlung the initial state of the electron must be represented by a plane wave plus an outgoing spherical wave, whereas the final state must be represented by an ingoing spherical wave. In this paper a theoretical justification for employing an ingoing modification of plane waves for final states in scattering is provided.
A. Salam (Cambridge, England).

Nordsieck, A. Reduction of an integral in the theory of bremsstrahlung. Physical Rev. (2) 93, 785–787 (1954).

A matrix-element integral for bremsstrahlung (or pair production), which occurs in the paper by Bethe and Maximon [see the third preceding review] is reduced to an ordinary hypergeometric function by contour integration methods. *A. Salam* (Cambridge, England).

Waldmann, Ludwig. Nichtrelativistische Quantenmechanik des starren Elektrons. Z. Naturforschung 8a, 583–593 (1953).

The non-relativistic theory of a rigid extended electron developed by Bopp and Hönl is applied to the Lamb shift in hydrogen, the Coulomb degeneracy being removed because of the finite size of the particle. The result depends correctly on Z and n, but its magnitude appears only on adjusting the one parameter of the theory, making the radius of the electron equal to ten times the classical radius. There does not appear to be any mechanism for describing the effects of vacuum polarization, which depend in such a different way on the mass of the orbital particle.
H. C. Corben (Pittsburgh, Pa.).

Úlehla, Ivan. Quantum mechanics of mesons with spin zero and one. Czechoslovak J. Phys. 3, 261–266 (1953). (Russian. English summary)

Assuming a Hamiltonian of the form $H=\alpha_r p^r+\mu\alpha_0$ and a minimal equation $(H^2-E^2)H=0$ imposes conditions on the matrices α_r which result in two cases: (i) Dirac's Hamiltonian for a particle of spin $\frac{1}{2}$, (ii) a Hamiltonian for a particle of spin at most 1. The latter is used to obtain explicit formulas for the coordinates of a free particle, which show a Zitterbewegung of a slightly more complex nature than that of a Dirac particle. *A. J. Coleman* (Toronto, Ont.).

Winogradzki, Judith. Contribution à la théorie des grandeurs physiques attachées aux particules de spin 1/2. Ann. Physique (12) 8, 763–812 (1953).

This dissertation derives from rather general considerations the known tensors of the zeroth, first and second order, which can be constructed from linear combinations of products of two components of a Dirac four spinor. Several quadratic relations among these various tensors are then obtained, a particular case being the Pauli-Kofink formulas.
A. J. Coleman (Toronto, Ont.).

Petiau, Gérard. Sur les solutions à singularités localisées dans le mouvement rectiligne et uniforme du corpuscule de spin $\hbar$. C. R. Acad. Sci. Paris 238, 1568–1570 (1954).

Rubinowicz, A. Propagation of a cut-off train of de Broglie waves. Acta Phys. Polonica 10, 79–86 (1950).

The author considers the one-dimensional propagation of the de Broglie waves of frequency ν_0 which at time $t=0$ are confined to the negative x axis, i.e.,

$$\psi=\exp\left[-2\pi i\nu_0(t-x/u_0)\right]$$

if $x<0$ and $\psi=0$ if $x>0$. Here u_0 is the phase velocity corresponding to the frequency ν_0. The subsequent motion is given by the integral

$$\psi(x,t)=\frac{1}{2\pi i}\int e^w\frac{d\nu}{\nu-\nu_0},\quad w=2\pi i\nu(t-x/u),$$

taken about the appropriate contour. He then shows that the wave appears at the point x (>0) at a time $t=x/c$ (c=light velocity), that this is followed by a "precursor" wave. The main body of the wave appears at a time such $t=x/v_0$, where $v_0u_0=c^2$. *H. Feshbach.*

Butcher, P. N. A variational formulation of the multi-stream electrodynamic field equations. Philos. Mag. (7) 44, 971–979 (1953).

The author uses the recent device of Dirac [Proc. Roy. Soc. London. Ser. A. 212, 330–339 (1952); these Rev. 14, 228] of introducing variables ξ and η specifying vortex lines of charge. He obtains a variational principle equivalent to an approximation to the Maxwell-Lorentz equations valid for velocities of the electric charge small compared with that of light. *A. J. Coleman* (Toronto, Ont.).

Thermodynamics, Statistical Mechanics

✶Sommerfeld, Arnold. Thermodynamik und Statistik. Herausgegeben von F. Bopp und J. Meixner. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1953. xiv+374 pp. (1 plate).

That this volume of lectures does not reach the absolute standard of the author's five others [Vorlesungen über theoretische Physik, Bd. I (3. Aufl., 1947), II (2. Aufl., 1949), VI (1. Aufl., 1947), Akademische Verlagsgesellschaft, Leipzig, Bd. III (1. Aufl., 1948), IV (1. Aufl., 1950), Dieterich'sche Verlagsbuchhandlung, Wiesbaden; these Rev. **10**, 195; for reviews of translations of vols. I, II, III, and VI see these Rev. **10**, 608; **11**, 700; **14**, 419, 433] is perhaps only to be expected: It deals with the hardest and most unruly subjects in classical physics. The other volumes attained logical order and precision without pretense at rigor in the sense of "advanced calculus"; deliberately they emphasized the rational side of their subjects, bringing in the physics as motivation or as illustration. The present volume makes a like effort, but success would have required greater revision of the common views than the author has attempted. While the author begins his preface with the statement that "Thermodynamics is the pattern and example of an axiomatic science," his development of it along classical lines presents the usual maze of undefined quantities, axioms used without statement, stated axioms that are not used, "theorems" that are not proved, paralogisms, circularities, and "approximations". However, in contrast to other writers on this subject the author often frankly admits that a "proof" is only a description in other words or that a "theorem" is really an axiom. The author describes Carathéodory's axiomatic development but does not use it; while he implies that it is too abstract, a more likely reason is that it suffices for only a small part of the field of thermodynamics presented in this book. The reader who seeks a physical treatment will find here a good one. Many details are treated with especial care, and probably this book is the best existing general introduction to the subjects concerned. Its greatest weakness is the number of different topics which it merely describes without developing, leaving the reader to look elsewhere for all the calculation. Misprints in references to earlier equations are frequent and often confusing. Next to its honesty and care, its greatest virtue is the many penetrating remarks and interconnections quoted from unexpected sources.

The first part, general thermodynamics, contains the principles and the central examples. It ends with Nernst's postulate. The second part presents brief descriptions of a large range of application. Included are the law of mass action, reversible mixing of gases, theory of dilute solutions, equilibrium of phases, electrochemical potentials, ferromagnetism and paramagnetism, cavity radiation. At the end is a section on the thermodynamics of irreversible processes, added by one of the editors. It is the clearest passage in the first half of the book.

In the principles of statistical mechanics the author maintains the conservative view that classical statistics must come first: Quantum statistics is mentioned briefly in §25B, §33, §36C, and §37–39.

Part III presents the elementary kinetic theory of gases. It includes not only the method of the mean free path but also simple theories of Brownian motion, paramagnetism, and the van der Waals equation. The material is not overemphasized, since it occupies only 35 pages.

Part IV presents statistical mechanics according to the enumeration method of Boltzmann. The editors have inserted a section describing the viewpoint of Gibbs, in which they claim that the assumptions of Gibbs and of Boltzmann are equivalent. As far as fundamental principles are concerned, Part IV is the weakest part of the book: The ergodic problem is passed over in silence, the work of Darwin and Fowler is so briefly described that the reader is unlikely to realize that a rigorous proof of Boltzmann's law is its object, and the fundamental work of Khinchin, who not only gave another rigorous proof but also by its means obtained a positive solution to the ergodic problem, is not mentioned. The reviewer regards most of Part IV as out of date in respect to principles. The emphasis in the author's and editors' treatment is on applications to problems of quantum physics.

Part V, "Outline of an exact kinetic theory of gases," is entirely the work of the editors. For a length of 45 pages this outline, in the reviewer's opinion, is unusually comprehensive and clear. The authors make use of the recent literature; for example, for the main theorem on collisional invariants they present the proof of H. Grad [Comm. Pure Appl. Math. **2**, 331–407 (1949); these Rev. **11**, 473]. While in their statements they attribute what seems to the reviewer unjustified value to the results of Chapman and Enskog (in the preface, Sommerfeld makes an equally unjustified reference to Hilbert), in the working they follow Grad in assuming for the distribution function an expansion in Hermite polynomials and in emphasizing the role of the moments. The calculations are somewhat more simply presented, and they use this apparatus to obtain viscosity and heat conductivity for a gas whose molecules are rigid spheres. The text ends with a discussion of an electron gas.

Appended to the book is a short set of exercises. About half are numerical, while half concern important additional topics whose mastery promotes understanding of the text. Unfortunately there is only one exercise for Part V.

The book is well set but ill bound. The paper is not likely to last. The frontispiece is a portrait of the author.

C. Truesdell (Bloomington, Ind.).

Popoff, Kyrille. Sur la thermodynamique des processus irréversibles. III. Z. Angew. Math. Physik **5**, 67–83 (1954).

[For parts I–II see same Z. **3**, 42–51, 440–448 (1952); these Rev. **13**, 808; **14**, 1047.] In connection with the phenomenological relations

$$(1)\qquad x_i' = \sum_k L_{ik} X_k,$$

where $X_i = -\partial(\Delta S)/\partial x_i$ and $\Delta S = -\frac{1}{2}\sum_{i,k} g_{ik} x_i x_k$, the following is shown. (1) If the x_i, and consequently the g_{ik}, are the only elements determining the state of a given adiabatic system, then the L_{ik} must be completely determined by the g_{ik}. The n constants of integration in the general integral of (1) are determined by the initial values of the x_i; but for this integral to represent an irreversible process, it is necessary that $x_i(+\infty) = 0$, and this imposes n new conditions on the L_{ik}. (2) All these conditions are met by the integrals of the system $d^2x_i/dt^2 = X_i$, where $X_i = -\partial(\Delta S)/\partial x_i$ in the most general case, the L_{ik} thus depending only on the g_{ik}. (3) These results are applied to the theory of the conduction of heat and the theory of phases.

C. C. Torrance.

de Groot, S. R., and Mazur, P. Extension of Onsager's theory of reciprocal relations. I. Physical Rev. (2) 94, 218–224 (1954).

Mazur, P., and de Groot, S. R. Extension of Onsager's theory of reciprocal relations. II. Physical Rev. (2) 94, 224–226 (1954).

The Onsager theory of irreversible thermodynamics is extended so as to be applicable to vector and tensor quantities as well as the usual scalar state variables. Reciprocal relations are then derived for heat conduction, diffusion, viscosity and cross-effects in the absence of an electromagnetic field. In Part II, the reciprocal relations for electrical conduction are derived. *G. Newell.*

Rocard, Yves. L'équation d'état des fluides d'après la théorie cinétique. Revue Sci. 90, 387–418 (1952).

The van der Waals equation of state for fluids is the result of an attempt to obtain an equation of relatively simple mathematical form that is in qualitative agreement with experiment and at the same time has a physical interpretation in terms of crude microscopic models of the fluid. This paper describes some of the failures of the van der Waals equation, both as regards its agreement with experiment and its theoretical justification. Consideration is given to various modifications that can be made to this equation in order to produce a more accurate equation and one having a stronger theoretical basis. *G. Newell.*

Ramakrishnan, Alladi. On the molecular distribution functions of a one-dimensional fluid. I. Philos. Mag. (7) 45, 401–409 (1954).

The properties of a one-dimensional fluid derived by Salsburg, Zwanzig and Kirkwood [J. Chem. Phys. 21, 1098–1107 (1953)] are obtained by considering the molecular distribution functions as product densities and using the properties of product densities derived by the author in a recent series of papers. *G. Newell* (Providence, R. I.).

Landsberg, Peter T. The continuous spectrum approximation in quantum statistics. Physical Rev. (2) 94, 469–471 (1954).

A certain class of quantum-mechanical systems of non-interacting particles is defined by imposing conditions on the asymptotic behavior of the energy levels for large volumes. It is shown that to calculate the mean value of an extensive variable, averaged over a canonical or grand canonical ensemble, one can use a continuous spectrum approximation and replace the sum over the discrete energy levels by an integral. Although many postulates are enumerated at the beginning, many others are made in the midst of the proof and several others, which would seem to be necessary, are not stated at all. *G. Newell.*

Landsberg, P. T. Quantum statistics of closed and open systems. Physical Rev. (2) 93, 1170–1171 (1954).

The behavior of a system of non-interacting Bose-Einstein or Fermi-Dirac particles is considered in the "limit L", a limit in which the number of particles N and the volume V approach infinity with N/V fixed. It is shown that the mean occupation numbers given by the grand canonical ensemble are the same as those of the canonical ensemble in the limit L if the average number of particles of the former is kept equal to the value of N for the latter.

The author claims to show this "whatever the system of energy levels". The reviewer feels that several important steps are left out and questions the generality of his conclusions for the following reasons. Several quantities defined for finite systems are considered in the limit L without justifying the existence of such limits. Furthermore, there is no mention anywhere regarding how any of these functions depend upon V even though limits as N and $V\to\infty$ are taken. Indeed V does not seem to enter the discussion in any very essential way. *G. Newell.*

Inagaki, M., Wanders, G., et Piron, C. Théorème H et unitarité de S. Helvetica Phys. Acta 27, 71–73 (1954).

Let $S(w)$ be a function (entropy) of the distribution w_i $(i=1, \cdots, n)$, which is symmetric in the w_i and has monotone decreasing derivatives with respect to the w_i. Let $A_{ik}\geqq 0$ be a matrix of transition probabilities with both the rows and columns normalized to 1; the A_{ik} are absolute squares of the elements of a unitary matrix. If $w''=\mathbf{A}w'$, the author states that $S(w'')-S(w')\geqq 0$ and proves this for the special case in which S is of the form $S=\sum_i f(w_i)$. *G. Newell* (Providence, R. I.).

Green, Melville S. Markoff random processes and the statistical mechanics of time-dependent phenomena. II. Irreversible processes in fluids. J. Chem. Phys. 22, 398–413 (1954).

In part I [same J. 20, 1281–1295 (1952); these Rev. 14, 1048] the author obtained equations describing the temporal behavior of any set of gross variables (macroscopic observables) which can be assumed to give a complete description of a macroscopic state. The present paper is concerned with special applications of these equations to the study of irreversible processes in fluids in which the gross variables are taken as some finite number of Fourier expansion coefficients of the mass, momentum, and energy distribution in space. From these coefficients one obtains a convenient representation of a "smoothed out" local density of mass, momentum, and energy.

Equations involving these gross variables are derived for one and two component systems and lead in the former case to the Navier-Stokes equations. The coefficients of viscosity, diffusion and heat conductivity are expressed in terms of autocorrelation coefficients of certain phase functions. Although it is almost impossible to obtain numerical evaluations of these coefficients for dense gases or liquids for which the theory is assumed to be valid, the author does show that for the rarefied gases, these coefficients agree with those given by the Chapman-Enskog theory. *G. Newell.*

Chester, G. V. The quantum-mechanical partition function. Physical Rev. (2) 93, 606–611 (1954).

The expansion of the quantum-mechanical partition function in powers of the interaction potential between particles, performed for Boltzmann statistics by H. S. Green [J. Chem. Phys. 19, 955–962 (1951)] and by Goldberger and Adams [ibid. 20, 240–248 (1952); these Rev. 13, 1013] using cumbersome operational methods, is here shown to be obtainable much more simply for all three statistics by a quite elementary method of expansion. It is also pointed out how limited the applicability of the expansion becomes when the interaction potential has singularities, as is usually the case in view of the incompressible cores of the particles. The modifications needed for the latter case are given. They require, however, an exact treatment of the hard-sphere problem. *L. Van Hove* (Utrecht).

Welander, Pierre. On the temperature jump in a rarefied gas. Ark. Fys. 7, 507–553 (1954).

When heat is carried away from a flat wall by conduction through an ideal gas, there exists at the wall a temperature difference between the wall and the adjacent gas and also a sort of boundary-layer region with a width of the order of a mean free path λ between the wall and the region where the temperature gradient is essentially constant. An attempt is made to analyse the behavior of the gas in this transition region and to obtain the temperature variation near the wall and particularly the difference between the wall temperature and the extrapolation to the wall of the linear temperature increase at large distances from the wall.

The analysis begins with the Boltzmann equation which is greatly simplified through the following principle approximations and assumptions. 1. The distribution function is nearly Maxwellian. One can write for $f(x, \mathbf{v})$, $f=f^{(0)}+f^{(1)}$ in which $f^{(0)}$ is a Maxwellian distribution with variable temperature and density. The Boltzmann equation is linearized by keeping only first order terms in $f^{(1)}$. 2. The collision integral is assumed to be proportional to $f^{(1)}$. This assumption is suggested partly by the theory of the Maxwell gas (inverse-fifth-power force law) but is a questionable one for the situation in which it is applied. The justification of this assumption is not discussed very much. It does simplify the Boltzmann equation to a very considerable extent, in fact, so much so that one can immediately obtain the general solution for $f^{(1)}$ in terms of $f^{(0)}$. 3. The molecules reflected from the wall have a Maxwellian distribution. The temperature of this distribution is not that of the wall but is a paramater that eventually is incorporated into an eigenvalue problem.

Having found $f^{(1)}$ in terms of $f^{(0)}$, the density $n(x)$ and temperature $T(x)$ of $f^{(0)}$ must be so adjusted that the boundary conditions imposed by 3 are satisfied and that the average of 1, v_x and v^2 over $f^{(1)}$ vanish. This gives a pair of linear integral equations for $n(x)$ and $T(x)$. The method parallels somewhat that of the first-order Chapman-Enskog theory except that a term $v_x\partial f^{(1)}/\partial x$ which is neglected in the first-order Chapman-Enskog theory is retained and in fact becomes the dominant term near the wall such as in the various boundary-layer theories. The result of the calculation is that the temperature jump differs from that given in the old (1898) Smoluchowski theory in that a factor $(2-a)/a$ now becomes $(2-ka)/a$; k is a constant found to be 0.827 and a is the accommodation coefficient. Although the paper is quite long, it does not contain a great deal of irrelevant details but does include a survey of most of the historical background necessary to understand the proposed developments. An appendix also outlines a corresponding treatment of viscous slip.

G. Newell (Providence, R. I.).

Welander, Pierre. Heat conduction in a rarefied gas: the cylindrically symmetrical case. Ark. Fys. 7, 555–564 (1954).

The conduction of heat by a gas between two concentric cylinders is analysed by appropriate modifications in the theory of the temperature jump at plane surfaces described in the preceding paper. The treatment is confined to the case in which the mean free path λ is small compared with the difference between the radii of the two cylinders but not necessarily small compared with the radius r_1 of the smaller cylinder. The problem is more difficult than the previous one and is not described in as much detail nor are the calculations as complete. The results are similar in form except the k of the previous paper becomes a function of r_1/λ which has not yet been found explicitly.

G. Newell.

Rosen, Philip. The solution of the Boltzmann equation for a shock wave using a restricted variational principle. J. Chem. Phys. 22, 1045–1049 (1954).

A method described as a "restricted variation principle" is used to find approximate solutions of the Boltzmann equation for a plane shock in a gas. The variation is "restricted" in the sense that the integral to be varied is a function of two unknown functions f and $\tilde{f}$ and the integral is required to be stationary with respect to small variations of f with $\tilde{f}$ fixed when $\tilde{f}\to f$. A trial function of a form proposed by Mott-Smith, a sum of two different Maxwellian distributions with amplitudes $n_\alpha(x)$ and $n_\beta(x)$, is substituted into this integral. A conservation of momentum for all x gives a relation between $n_\alpha(x)$ and $n_\beta(x)$ and the other relation required to determine $n_\alpha(x)$ and $n_\beta(x)$ is found by requiring the integral to be stationary with respect to variations of $n_\alpha(x)$ according to this special prescription.

This procedure gives a best solution, in some sense, from a set of functions of a given functional form. The results are compared with those of Mott-Smith [Physical Rev. (2) 82, 885–892 (1951); these Rev. 12, 891] who used a different criterion for a best solution. It is shown that the errors in the approximation are on the average smaller for shocks of higher Mach numbers. It is not obvious that this variation principle has a unique exact solution or what significance is to be attached to this best approximate solution of given form even though the latter seems to be well defined, at least in the case considered.

G. Newell.

Bayet, Michel, Delcroix, Jean-Loup, et Denisse, Jean-François. Sur la résolution de l'équation de Boltzmann dans le cas d'un gaz de Lorentz; application aux gaz faiblement ionisés. C. R. Acad. Sci. Paris 238, 2146–2148 (1954).